

The Economics of Hedge Funds*

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March 22, 2012

Abstract

Hedge fund managers are often compensated via management fees on the assets under management (AUM) and incentive fees indexed to the high-water mark (HWM). We develop an analytically tractable model of hedge fund leverage and valuation where the manager maximizes the present value (PV) of future management and incentive fees from current and future managed funds. By leveraging on an alpha strategy, a skilled manager can create significant value for investors. However, leverage also increases the likelihood of poor performances, which may trigger money outflow, withdraw/redemption, and forced fund liquidation, causing the manager to lose future fees. Intuitively, the ratio between AUM and HWM, w , measures the optionality of the long position in incentive fees and the short position in investors' liquidation and redemption options. Moreover, w determines the manager's optimal leverage and dynamic valuation of fees.

Our main results are (1) managerial concern for fund survival induces the manager to choose prudent leverage; (2) leverage depends on w and tends to increase following good performances; (3) both incentive and management fees contribute significantly to the manager's total value; (4) performance-triggered new money inflow encourages the manager to increase leverage and has large effects on the manager's value, particularly on the value of incentive fees; (5) fund restart and HWM reset options are valuable for the manager; (6) managerial ownership has incentive alignment effects; (7) when liquidation risk is low, the manager engages in risk seeking and the margin requirement/leverage constraint binds. Our framework allows us to infer the minimal level of managerial skill, the un-levered break-even alpha, demanded by investors in a competitive equilibrium for a given managerial compensation contract.

*First Draft: May 2010. We thank Andrew Ang, George Aragon, Patrick Bolton, Pierre Collin-Dufresne, Kent Daniel, Darrell Duffie, Will Goetzmann, Rick Green, Jim Hodder, Bob Hodrick, Jason Hsu, Tim Johnson, Lingfeng Li, Bing Liang, Stavros Panageas, Lasse Pedersen, Chester Spatt, René Stulz, Suresh Sundaresan, Sheridan Titman, Laura Vincent, Jay Wang, Zhenyu Wang, Mark Westerfield, and seminar participants at ASU, Brock, CMU Tepper, Columbia, McGill, New York Fed, UIUC, PREA, SUFE for helpful comments.

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1 Introduction

Hedge funds' management compensation contracts typically feature both management fees and performance/incentive fees. The management fee is charged as a fraction, e.g. 2%, of assets under management (AUM). The incentive fee, a key characteristic that differentiates hedge funds from mutual funds, is calculated as a fraction, e.g. 20%, of the fund's profits. The cost base for the profit calculation is often investors' high-water mark (HWM), which effectively keeps track of the maximum value of the invested capital and critically depends on the manager's dynamic investment strategies. This compensation scheme is often referred to as the "two-twenty" contract. An important feature of hedge funds is the sophisticated use of leverage. Hedge funds may borrow through the repo markets or from prime brokers, as well as use various implicit leverage, often via options and other derivatives.

We develop an integrated dynamic framework of leverage and valuation for hedge funds. By leveraging on the alpha-generating strategy, a skilled manager creates value for investors. However, leverage also increases the likelihood of poor performances. In practice, a fund that performs poorly often faces money outflow, withdraw/redemption, or liquidation. We model performance-triggered fund liquidation via a liquidation boundary. Upon liquidation, the manager may lose future fees. The manager dynamically chooses leverage to maximize the present value (PV) of all fees from both the current and future managed funds. Outside investors rationally participate in the fund given their beliefs about the managerial skills and leverage strategies.

Specifically, our model contains the following important features: (1) an alpha generating strategy; (2) management fees as a fraction of the AUM; (3) incentive/performance fees linked to the HWM; (4) poor performance-triggered liquidation; (5) margin requirement or leverage constraint; (6) managerial ownership, which is often motivated as an incentive alignment mechanism; (7) performance-induced new money inflow; (8) the manager's option to restart a fund (endogenous managerial outside option) at a cost. To simplify exposition, in our baseline model, we incorporate the first four features and focus on the manager's key tradeoff between the value creation benefit and the liquidation cost due to leverage. We then introduce each new feature from (5) to (8) individually into our baseline model and analyze their economic and quantitative implications.

Our model is analytically tractable. We obtain closed-form solutions up to an ordinary

differential equation (ODE). The key state variable, denoted by w , is the ratio between the fund's AUM and its HWM. In a dynamic framework with liquidation or other downside risks, the risk-neutral manager has incentives to preserve the fund's going-concern value so as to collect fees in the future. The risk-neutral manager's precautionary motive induces risk-averse managerial behavior. The manager's optimal leverage increases with alpha, decreases with variance and the manager's effective risk aversion, similar to the risk-averse investor's portfolio allocation as in Merton (1971). Importantly, unlike the Merton-type investor, both the manager's effective risk aversion and leverage are stochastic and depend on w . The closer the fund's AUM is to its HWM (the higher w), the less likely the fund is liquidated and the more likely the manager collects the incentive fees, the less risk averse the manager behaves and the higher the leverage.

While our baseline model parsimoniously captures the key tradeoff between value creation and costly liquidation, such a simple framework inevitably misses some key features of hedge funds which may have first-order effects. We introduce new features, (5)-(8) listed previously, one at a time into the baseline model to highlight their implications. First, when liquidation risk is low, the manager can be risk seeking and the margin requirement becomes binding. Second, managerial ownership within the fund mitigates agency conflicts. Third, we incorporate money flow-performance relation and find that it has significant implications on the manager's leverage choices and the PVs of management fees and of incentive fees. Finally, we integrate the manager's options to start up new funds and/or to reset the fund's HWMs in our framework and find that these options are quantitatively valuable.

To provide quantitative implications, we calibrate our baseline model using empirical leverage moments reported in Ang, Gorovyy, and van Inwegen (2011). Our calibration suggests that the manager collects about 20 cents for each dollar under management in the PV sense. Under competitive equilibrium, investors break even in PV, and the manager generates 20% surplus on the AUM and captures all the surplus via their compensation. Out of the manager's total value creation of 20 cents on a dollar, 75% is attributed to management fees (15 cents) and the remaining 25% is delivered via incentive fees (5 cents). By incorporating managerial risk-seeking incentives/margin requirements, managerial ownership, new money inflow, and fund restart/HWM reset options, the manager has additional incentives to lever, which in turn makes the value of incentive fees more important. Quantitatively,

management fees remain a significant and often majority contributor to the manager's total value. Metrick and Yasuda (2010) find that management fees are also quantitatively important in private equity where funds managers also charge management and incentive fees via two-twenty-type compensation contracts. While the compensation structure is similar in essence, institutional details such as how management fees and performance fees are calculated differ significantly for hedge funds and private equity funds.

Related literature. There are only a few theoretical papers studying the hedge fund's valuation and leverage decisions. Goetzman, Ingersoll, and Ross (2003), henceforth GIR, provide the first quantitative inter-temporal valuation framework for management and incentive fees in the presence of the HWM. They derive closed-form solutions for both investors' and the manager's values for a fund with a constant alpha. GIR focuses solely on valuation and does not allow for managerial leverage or any other decisions such as fund closure/restart.

Panageas and Westerfield (2009), henceforth PW, study the effects of HWM-based incentive fees on managerial leverage choice and valuation when the manager is compensated solely via incentive fees. As in PW, we also endogenize leverage but we introduce two new critical features, management fees and liquidation/redemption risk into our baseline model. We show that the two new features as well as performance fees all play essential roles on leverage and the manager's value. Unlike PW which generates constant leverage at all times and for all values of w , the ratio between the AUM and the HWM, our model generates time-varying leverage which critically depends on w . In our model, the manager's value decreases with HWM and increases with incentive fees; both predictions are opposite of theirs. Quantitatively, we show that both incentive and management fees are important contributors to the manager's total value, while the manager's value in their model solely comes from incentive fees by construction. When liquidation risk is low, the manager becomes risk seeking and the margin requirement becomes binding. Finally, we also incorporate managerial ownership, new money inflow, and managerial fund restart or HWM reset options, and find that the quantitative implications of these features can be quite significant.

Hodder and Jackwerth (2007) numerically solve a risk-averse manager's investment strategy in a discrete-time finite-horizon model. Our work also relates to Dai and Sundaresan (2010), which point out that the hedge fund is short in two important options, investors' redemption option and funding options (from prime brokers and short-term debt markets),

and the hedge fund’s short positions in these two options have significant effects on the fund’s risk management policies.

2 Model

In this section, we set up a model with essential but minimal ingredients.

The fund’s investment opportunity. The manager can always invest in the risk-free asset which pays interest at a constant rate r . Additionally, the manager generates risk-adjusted expected excess returns, referred to as alpha. Without leverage, the incremental return for the skilled manager’s alpha-generating investment strategy, dR_t , is given by

$$dR_t = (r + \alpha) dt + \sigma dB_t, \tag{1}$$

where B is a standard Brownian motion, α denote the expected return in excess of the risk-free rate r , and σ is the return volatility. Alpha measures scarce managerial talents, which earn rents in equilibrium (Berk and Green (2004)). As we will show later, even with time-invariant investment opportunity, the optimal leverage will change over time due to managerial incentives and liquidation risk.

Let W be the fund’s AUM and D denote the amount invested in the risk-free asset. The investment amount in the alpha strategy (1) is then $W - D$. Let π denote the (risky) asset-capital ratio, $\pi = (W - D)/W$. Hedge funds often borrow via short-term debt and obtain leverage from the fund’s prime brokers, repo markets, and the use of derivatives.¹ For a levered fund, $D < 0$ and $\pi > 1$. For a fund hoarding cash, $D > 0$ and $0 < \pi < 1$.

Managerial compensation contracts. Managers are paid via both management and incentive fees. The management fee is specified as a constant fraction c of the AUM W , $\{cW_t : t \geq 0\}$. The incentive fee often directly links compensation to the fund’s performance via the so-called high-water mark (HWM).

When the AUM W exceeds the HWM H , the manager collects a fraction k of the fund’s performance exceeding its HWM and then resets the HWM H . We introduce the dynamics of the HWM H on the boundary $W = H$ in Section 3.

¹Few hedge funds are able to directly issue long-term debt or secure long-term borrowing.

When the AUM W is below the HWM H , the HWM may still evolve due to indexed growth or investors' withdrawal. Let g denote the growth rate of H in the absence of payout. This growth rate g may be set to zero, the interest rate r , or other benchmarks. As in GIR, investors in our model are paid continuously at a rate δW_t where $\delta \geq 0$ is a constant. Naturally, the fund's HWM is adjusted downward to account for the payout rate δ . To summarize, when $W < H$, the HWM H evolves deterministically as follows,

$$dH_t = (g - \delta)H_t dt, \quad \text{if } W_t < H_t. \quad (2)$$

When $g = \delta$, (2) implies that the HWM H is the running maximum of W , $H_t = \max_{s \leq t} W_s$, in that the HWM is the highest level that the AUM has attained.

Fund liquidation. As in GIR, the fund can be exogenously liquidated with probability λ per unit of time. By assumption, the manager can do nothing to influence this liquidation likelihood. Let τ_1 denote the exogenous stochastic liquidation time.

Next, we turn to endogenous liquidation. With an alpha strategy and no frictions, the model features constant return to scale with an unbounded alpha generating technology (via leverage). Therefore, the optimal leverage for a risk-neutral manager would be infinite without frictions, an economically unrealistic case. High leverage implies highly volatile AUM and hence potentially very large losses for the fund. Large losses may cause investors to lose confidence in the manager, triggering liquidation.

Additionally, conflicts of interest and incomplete contracts within institutions (clients) potentially cause institutional clients to liquidate their positions in the hedge fund even if they believe that the hedge fund manager has alpha but is simply unlucky. For example, pension fund managers may involuntarily liquidate investments in the hedge fund due to their career concerns or simply the difficulty to convince retirees or the pension fund board that the selected hedge fund manager who just incurred a huge loss is unlucky, but skilled.

We assume that the fund's sufficiently disappointing performance triggers liquidation. Specifically, when the AUM W falls to a fraction b of its HWM H , the fund is liquidated. For example, if $b = 0.7$, the fund will be liquidated if the manager loses 30% of the AUM from its HWM. GIR makes a similar liquidation assumption in their valuation model. Unlike GIR, the AUM dynamics in our model are endogenous. Let τ_2 denote this endogenous

performance-triggered stochastic liquidation time,

$$W_{\tau_2} = bH_{\tau_2}. \quad (3)$$

The above liquidation condition has been used by Grossman and Zhou (1993) in their study of investment strategies by institutional investors facing what they refer to as “draw-down” constraints. In their terminology, $1 - b$ is the maximum “drawdown” that investors allow the fund manager before liquidating the fund in our model. Grossman and Zhou (1993) state that “the authors’ knowledge and experience of the area of investment management where leverage is used extensively (such as the trading of futures, options, and foreign exchange) has convinced us that an essential aspect of the evaluation of investment managers and their strategies is the extent to which large drawdowns occur. It is not unusual for such managers to be fired subsequent to achieving a large drawdown, nor is it unusual for the managers to be told to avoid drawdowns larger than 25%.”

The fund is liquidated either exogenously at stochastic time τ_1 or endogenously at τ_2 . At liquidation time $\tau = \tau_1 \wedge \tau_2$, the manager receives nothing and investors collect the fund’s AUM W_τ . While leveraging on an alpha strategy creates value, the manager is averse to losing all future fees upon liquidation. As a result, from investors’ perspective, the manager effectively under-invests by choosing a conservative leverage in order to preserve future management and incentive fees. The manager chooses the fund’s leverage to balance the benefits of leveraging against the cost of liquidation.

Dynamics of AUM. The fund’s AUM W_t evolves as follows

$$\begin{aligned} dW_t = & \pi_t W_t (\mu dt + \sigma dB_t) + (1 - \pi_t) r W_t dt - \delta W_t dt - c W_t dt \\ & - k [dH_t - (g - \delta) H_t dt] - dJ_t, \quad t < \tau. \end{aligned} \quad (4)$$

The first and second terms in (4) describe the change of AUM W given the manager’s leverage strategy. The third term gives the continuous payout to investors. The fourth term gives the flow of management fees (e.g. $c = 2\%$), and the fifth term gives the incentive/performance fees which is paid if and only if the AUM exceeds the HWM (e.g. $k = 20\%$).² The process J in the last (sixth) term is a pure jump process which describes the purely exogenous

²The manager collects the incentive fees if and only if $dH_t > (g - \delta) H_t dt$.

liquidation risk: The AUM W is set to zero when the fund is exogenously liquidated. This pure jump process occurs with probability λ per unit of time.

Various value functions for investors and the manager. We now define various present values (PVs). Let $M(W, H; \pi)$ and $N(W, H; \pi)$ denote the PVs of management and incentive fees, respectively, for a given dynamic leverage strategy π ,

$$M(W, H; \pi) = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} c W_s ds \right], \quad (5)$$

$$N(W, H; \pi) = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} k [dH_s - (g - \delta) H_s ds] \right]. \quad (6)$$

We assume that the manager collects neither management nor incentive fees after stochastic liquidation.³ Let $F(W, H; \pi)$ denote the PV of total fees, which is given by

$$F(W, H; \pi) = M(W, H; \pi) + N(W, H; \pi). \quad (7)$$

Similarly, we define investors' value $E(W, H)$ as follows

$$E(W, H; \pi) = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} \delta W_s ds + e^{-r(\tau-t)} W_\tau \right]. \quad (8)$$

In general, investors' value $E(W, H)$ is different from the AUM W because of managerial skills. The total PV of the fund $V(W, H)$ is given by the sum of $F(W, H)$ and $E(W, H)$:

$$V(W, H; \pi) = F(W, H; \pi) + E(W, H; \pi). \quad (9)$$

The manager's optimization and investors' participation. Anticipating that the manager behaves in self interest, investors rationally demand that the PV of their payoffs, $E(W, H)$, is at least higher than their time-0 investment W_0 in order to break even in PV. At time 0, by definition, we have $H_0 = W_0$. Thus, at time 0, we require

$$E(W_0, W_0; \pi) \geq W_0. \quad (10)$$

Intuitively, the surplus that investors collect depends on their relative bargaining power versus the manager. In perfectly competitive markets, the skilled manager collects all the

³We may allow the manager to close the fund and start a new fund. The manager then maximizes the PV of fees from the current fund and the "continuation" value from managing future funds. See Section 9 for this extension.

surplus, and the above participation constraint (10) holds with equality. However, in periods such as a financial crisis, investors may earn some rents by providing scarce capital to the manager and hence investors' participation constraint (10) may hold with slack.

The manager dynamically chooses leverage π to maximize the PV of total fees,

$$\max_{\pi} F(W, H; \pi), \quad (11)$$

subject to the liquidation boundary (3) and investors' voluntary participation (10).

3 Solution

The manager maximizes the PV of total fees by trading off the benefit of leveraging on the alpha strategy against the increased liquidation risk due to leverage. We consider the parameter space where leverage-induced liquidation risk is sufficiently significant so that the manager's optimal leverage management problem is well defined and admits an interior leverage solution.⁴ We show that optimal leverage is time-varying. The manager de-levers as the fund gets close to the liquidation boundary in order to lower the fund's volatility, preserve the fund as a going-concern so that the manager continues to collect fees.

The interior region ($W < H$). In this region, we have the following Hamilton-Jacobi-Bellman (HJB) equation,

$$(r + \lambda)F(W, H) = \max_{\pi} cW + [\pi\alpha + (r - \delta - c)]WF_W(W, H) + \frac{1}{2}\pi^2\sigma^2W^2F_{WW}(W, H) + (g - \delta)HF_H(W, H). \quad (12)$$

The first term on the right side of (12) gives the management fee, cW . The second and third terms give the drift (expected change) and the volatility effects of the AUM W on $F(W, H)$, respectively. Finally, the last term on the right side of (12) describes the effect of the HWM H change on $F(W, H)$. Note that there is no volatility effect from the HWM H because H is locally deterministic in the interior region. The left side of (12) elevates the discount rate from the interest rate r to $(r + \lambda)$ to reflect the exogenous stochastic liquidation likelihood.

The HJB equation (12) implies the following first-order condition (FOC) for leverage π :

$$\alpha WF_W(W, H) + \pi\sigma^2W^2F_{WW}(W, H) = 0. \quad (13)$$

⁴Otherwise, the benefit of leveraging is too large and may cause value to be infinity, an unrealistic case.

The FOC (13) characterizes the optimal leverage, when $F(W, H)$ is concave in W , equivalently stated, the second-order condition (SOC) is satisfied. When the SOC is violated, we need additional constraints to ensure that the firm has a finite optimal leverage, as we show in Section 6. Next, we turn to the behavior at the boundary $W = H$.

The upper boundary ($W = H$). Our reasoning for the boundary behavior essentially follows GIR and Panageas and Westerfield (2009). A positive return shock increases the AUM from $W = H$ to $H + \Delta H$. The PV of total fees for the manager is then given by $F(H + \Delta H, H)$ before the HWM adjusts. Immediately after the positive shock, the HWM adjusts to $H + \Delta H$. The manager collects the incentive fees $k\Delta H$, and consequently the AUM is lowered from $H + \Delta H$ to $H + \Delta H - k\Delta H$. The PV of total fees is $F(H + \Delta H - k\Delta H, H + \Delta H)$. Using the continuity of $F(W, H)$ before and after the adjustment of the HWM, we have

$$F(H + \Delta H, H) = k\Delta H + F(H + \Delta H - k\Delta H, H + \Delta H). \quad (14)$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$kF_W(H, H) = k + F_H(H, H). \quad (15)$$

The above is the value-matching condition for the manager on the boundary $W = H$. By using essentially the same logic, we obtain the following boundary conditions for the PV of management fees $M(W, H)$ and the PV of incentive fees $N(W, H)$ at the boundary $W = H$: $kM_W(H, H) = M_H(H, H)$ and $kN_W(H, H) = k + N_H(H, H)$.

The lower liquidation boundary ($W = bH$). At the liquidation boundary $W = bH$, the manager loses all future fees in our baseline model, in that

$$F(bH, H) = 0. \quad (16)$$

This assumption is as the one in GIR. However, unlike GIR, the manager influences the liquidation likelihood via dynamic leverage. Recall that without this liquidation boundary ($b = 0$), the risk-neutral manager will choose infinite leverage and the manager's value is unbounded. By continuity, we require b to be sufficiently high so that the manager is sufficiently concerned about the liquidation risk and thus prudently manages leverage. In Section 9, we extend our model to allow the manager to start a new fund, enriching the baseline model by providing the manager with exit options.

The homogeneity property. Our model has the homogeneity property: If we double the AUM W and the HWM H , the PV of total fees $F(W, H)$ will correspondingly double. The effective state variable is therefore the ratio between the AUM W and the HWM H , $w = W/H$. We use the lower case to denote the corresponding variable in the upper case scaled by the contemporaneous HWM H . For example, $f(w) = F(W, H)/H$.

Summary of main results. With sufficiently high liquidation boundary b , the optimization program converges and optimal leverage is finite and time-varying. Using the homogeneity property to simplify the FOC (13), we obtain the following formula for leverage:

$$\pi(w) = \frac{\alpha}{\sigma^2 \psi(w)}, \quad (17)$$

where $\psi(w)$ is given by

$$\psi(w) = -\frac{w f''(w)}{f'(w)}. \quad (18)$$

With sufficiently large liquidation risk, the risk-neutral manager behaves in a risk-averse manner, which implies that the manager's value function $f(w)$ is concave. Therefore, we may naturally interpret $\psi(w) = -w f''(w)/f'(w)$ as the manager's "effective" relative risk aversion, which is analogous to the definition of risk aversion for a consumer.

Analogous to Merton's mean-variance portfolio allocation rule, optimal leverage $\pi(w)$ is given by the ratio between (1) the excess return α and (2) the product of variance σ^2 and risk aversion $\psi(w)$. However, unlike Merton (1971), the manager in our model is risk neutral, the curvature of the manager's value function and the implied stochastic effective relative risk aversion $\psi(w)$ are caused by the endogenous liquidation risk. Using the optimal leverage rule and Ito's formula, we may write the dynamics for $w = W/H$ as follows,

$$dw_t = [\pi(w_t)\alpha + r - g - c]w_t dt + \sigma \pi(w_t)w_t dB_t - dJ_t, \quad (19)$$

where the optimal leverage $\pi(w)$ is given by (17) and J is the pure jump process leading to liquidation, as we have previously described.

The manager's value $f(w)$ solves the following ordinary differential equation (ODE),

$$(r - g + \delta + \lambda)f(w) = cw + [\pi(w)\alpha + r - g - c]w f'(w) + \frac{1}{2}\pi(w)^2 \sigma^2 w^2 f''(w), \quad (20)$$

subject to the following boundary conditions,

$$f(b) = 0, \tag{21}$$

$$f(1) = (k + 1)f'(1) - k. \tag{22}$$

Additionally, investors' voluntary participation condition (10) can be simplified to

$$e(1) \geq 1. \tag{23}$$

Equation (21) states that the manager's value function is zero at the liquidation boundary b . Equation (22) gives the condition at the right boundary $w = 1$. Using the optimal leverage $\pi(w)$, we may calculate the PV of management fees $m(w)$, the PV of incentive fees $n(w)$, investors' payoff $e(w)$, and the total fund value $v(w)$. The appendix contains the details.

4 Leverage and the manager's value

We first choose the parameter values and then analyze the model's results.

4.1 Parameter choices and calibration

As in GIR, our model identifies $\delta + \lambda$, the sum of payout rate δ and the fund's exogenous liquidation intensity λ . We refer to $\delta + \lambda$ as the total withdrawal rate. Similarly, our model identifies $r - g$, which we refer to as the net growth rate of w (without accounting for fees). Our model is parsimonious and we only need to choose the following parameter values: (1) the un-levered α , (2) the un-levered volatility σ , (3) the management fee c and the incentive fee k , (4) the total withdrawal rate, $\delta + \lambda$, (5) the net growth rate of w , $r - g$, and (6) the liquidation boundary b . All rates are annualized and continuously compounded, when applicable.

We choose the commonly used 2-20 compensation contract, $c = 2\%$ and $k = 20\%$. We set the net growth rate of w to zero, $r - g = 0$. Otherwise, even unskilled managers will collect incentive fees by simply holding a 100% position in the risk-free asset. We set the exogenous liquidation probability $\lambda = 10\%$ so that the implied average fund life (with exogenous liquidation risk only) is ten years. Few hedge funds have regular payouts to investors, we choose $\delta = 0$. The total withdrawal rate is thus $\delta + \lambda = 10\%$.

We now calibrate the remaining three parameters: excess return α , volatility σ , and the liquidation boundary b . We use two moments from Ang, Gorovyy, and van Inwegen (2011), which report that the average long-only leverage is 2.13 and the standard deviation for cross-sectional leverage is 0.616 (for a data-set from a fund-of-hedge funds). Calibrating to the two leverage moments and the equilibrium condition, $e(1) = 1$, we identify $\alpha = 1.22\%$, $\sigma = 4.26\%$, and $b = 0.685$. The implied Sharpe ratio for the alpha strategy is $\eta = \alpha/\sigma = 29\%$.

Our calibration-implied maximum drawdown before investors liquidate the fund (or equivalently fire the manager) is $1 - b = 31.5\%$. Interestingly, this calibrated value 31% is comparable to the drawdown level of 25% that is quoted by Grossman and Zhou (1993) in their study of investment strategy with drawdown constraints.

4.2 Leverage π , managerial value $f(w)$, and risk aversion $\psi(w)$

Dynamic leverage. Figure 1 plots leverage $\pi(w)$ for $b \leq w \leq 1$. Leverage $\pi(w_t)$ is stochastic and time-varying. At the liquidation boundary $b = 0.685$, the fund is barely levered, $\pi(b) = 1.03$. At the upper boundary $w = 1$, leverage reaches $\pi(1) = 3.18$. For our calibration, as w increases, the manager increases leverage. The higher the value of w , the closer the manager is to collecting incentive fees and the more distant the fund is from liquidation, incentive fees become deeper in the money, and the higher leverage $\pi(w)$.

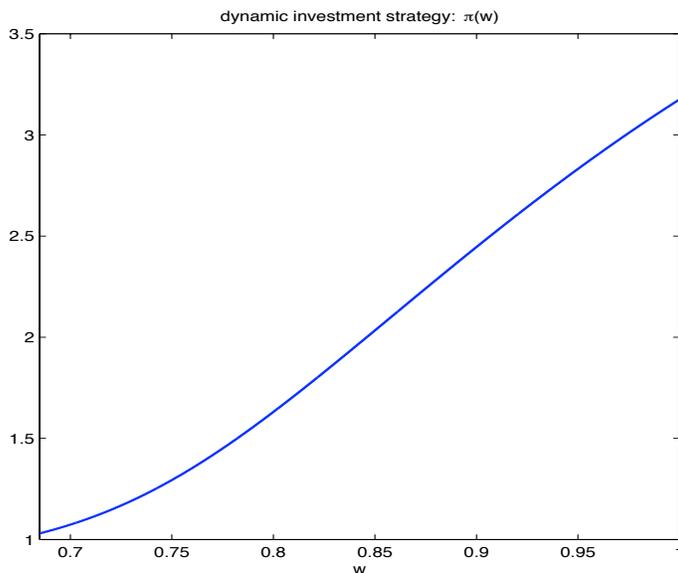


Figure 1: **Dynamic leverage** $\pi(w)$.

The manager’s value $f(w)$ and managerial risk aversion $\psi(w)$. With sufficiently high performance-triggered liquidation risk, $f(w)$ is concave in w . Panel A of Figure 2 plots $f(w)$. Quantitatively, for each unit of AUM, the manager creates 20% surplus in PV, $f(1) = 0.20$, and collect the surplus via management and incentive fees. Panel B of Figure 2 plots the risk-neutral manager’s “effective” risk aversion, $\psi(w)$. At the liquidation boundary $b = 0.685$, $\psi(b) = 6.50$, which is much larger than $\psi(1) = 2.11$, the manager’s risk aversion at $w = 1$. The manager’s effective risk aversion $\psi(w)$ is stochastic and ranges from 2 to 6.5, which is comparable to the typical values for the coefficient of relative risk aversion.

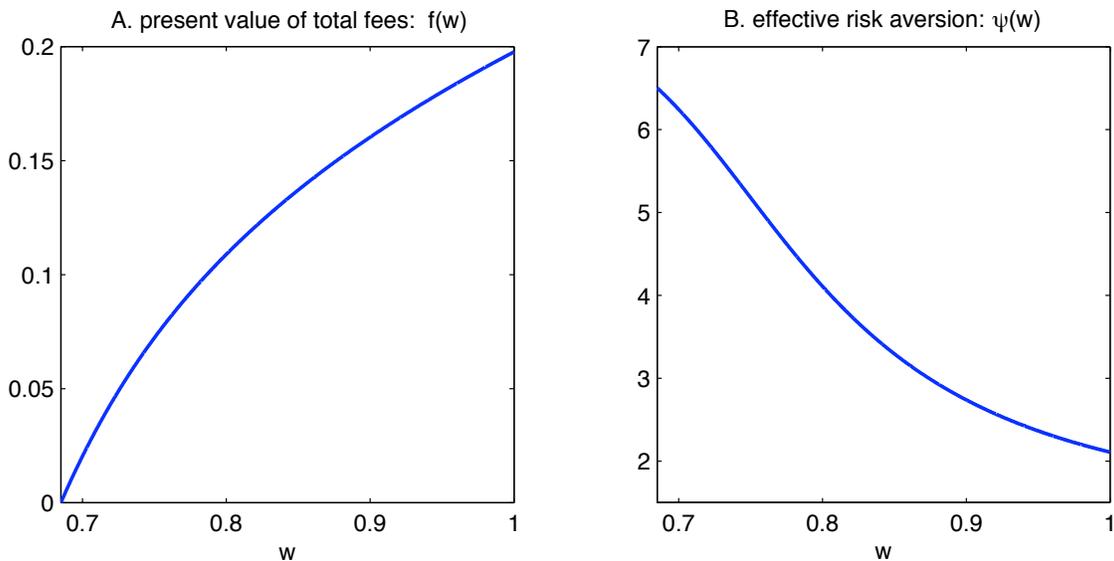


Figure 2: **The manager’s scaled value function $f(w)$ and the “effective” risk aversion, $\psi(w) = -w f''(w)/f'(w)$.**

The risk-neutral manager behaves in a risk-averse manner in our model because of aversion to inefficient (and hence costly) fund liquidation. For our calibration, as w increases, liquidation risk decreases and managerial risk aversion $\psi(w)$ falls.

4.3 Marginal effects of AUM and HWM on manager’s value

The marginal value of the AUM W , $F_W(W, H)$. The homogeneity property implies that the marginal value of the AUM is $F_W(W, H) = f'(w)$. Panel A of Figure 3 plots $f'(w)$. At the liquidation boundary $b = 0.685$, $f'(b) = 1.46$, which implies that the manager receives

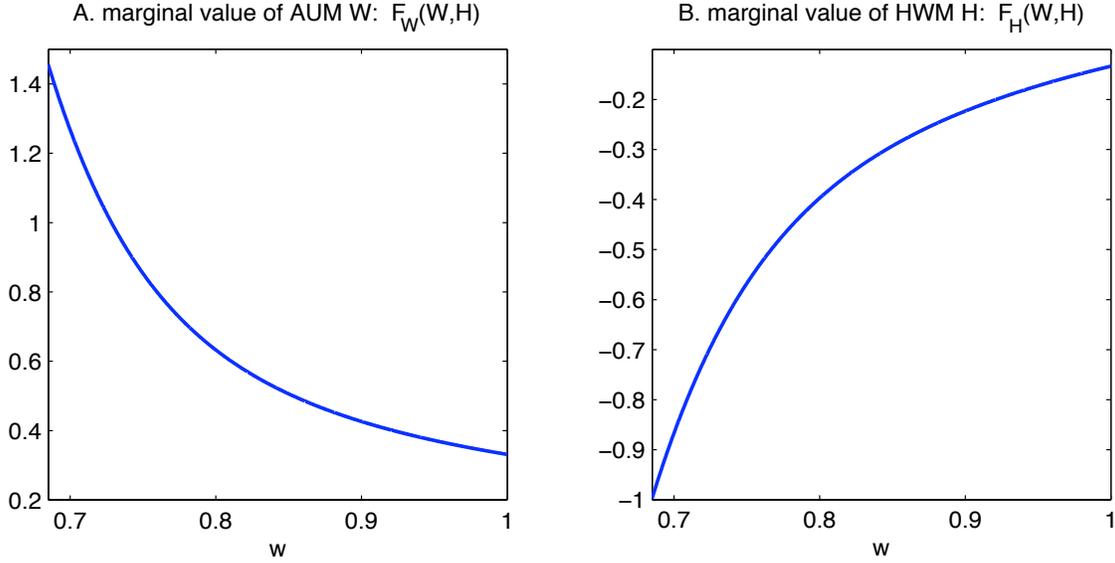


Figure 3: **The sensitivities of the manager's value function $F(W, H)$ with respect to the AUM W and the HWM H , $F_W(W, H)$ and $F_H(W, H)$.**

1.46 in PV for an incremental unit of AUM at $w = b$. As w increases, $f'(w)$ decreases. At $w = 1$, $f'(1) = 0.33$, which is only 23% of $f'(b) = 1.46$. Intuitively, a dollar of AUM near the liquidation boundary b is much more valuable than a dollar near $w = 1$ because the former decreases the risk of fund liquidation and can potentially save the fund from liquidation. The higher the value of w , the lower the liquidation risk and thus the lower the marginal value of AUM $f'(w)$.

The marginal impact of the HWM H , $F_H(W, H)$. Using the homogeneity property, we have $F_H(W, H) = f(w) - wf'(w)$. Panel B of Figure 3 plots $F_H(W, H)$ as a function of w . Increasing H mechanically lowers $w = W/H$, which reduces the value of incentive fees and increases the likelihood of investors' liquidation. Because the manager is long in incentive fees and short in the liquidation option, increasing H lowers $F(W, H)$, $F_H(W, H) < 0$.

Quantitatively, the impact of the HWM H on $F(W, H)$ is significant, especially when w is near the liquidation boundary. Because $F_H(W, H) < 0$ and $dF_H/dw = -wf''(w) > 0$ due to the concavity of $f(w)$, the impact of HWM H on the manager's total value $F(W, H)$ is smaller when the value of w is higher.

Even when the manager is very close to collecting incentive fees ($w = 1$), a unit increase

of the HWM H lowers the manager's value $F(H, H)$ by 0.13, which follows from $F_H(H, H) = f(1) - f'(1) = -0.13$. The impact of H on $F(W, H)$ is even greater for lower values of w . At the liquidation boundary $b = 0.685$, the impact of HWM on manager's value is about one to one in our calibration, $F_H(bH, H) = f(b) - bf'(b) = -1.00$. Intuitively, for a given value of W , increasing H moves the fund closer to liquidation and lowers the fund's going-concern value. The closer the fund is to liquidation, the more costly a unit increase of the HWM H .

In a model with incentive fees only, Panageas and Westerfield (2009) show that the manager's value function increases with the HWM H opposite to ours. Next, we value the manager's incentive and management fees.

5 Valuing incentive and management fees

In this section, we calculate the PV of management fees $m(w)$ and the PV of incentive fees $n(w)$, and then assess their contributions to the manager's total value $f(w)$. First, we sketch out the valuation formulas. The value functions $M(W, H)$, $N(W, H)$, $E(W, H)$, and $V(W, H)$ are all homogeneous with degree one in AUM W and HWM H . Therefore, we will use their respective values scaled by HWM H . The lower case maps to the variable in the corresponding upper cases, e.g. $M(W, H) = m(w)H$ and $N(W, H) = n(w)H$.

Valuation formulas. The scaled values $m(w)$ and $n(w)$ solve the following ODEs,

$$(r - g + \delta + \lambda)m(w) = cw + [\pi(w)\alpha + r - g - c]m'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2m''(w), \quad (24)$$

$$(r - g + \delta + \lambda)n(w) = [\pi(w)\alpha + r - g - c]n'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2n''(w), \quad (25)$$

with the following boundary conditions

$$m(b) = n(b) = 0, \quad (26)$$

$$m(1) = (k + 1)m'(1), \quad (27)$$

$$n(1) = (k + 1)n'(1) - k, \quad (28)$$

We next explore the quantitative implications of valuation formulas. Figure 4 plots $n(w)$ and $m(w)$ and their sensitivities $n'(w)$ and $m'(w)$.

The value of incentive fees $n(w)$. Panel A plots $n(w)$. By assumption, at the liquidation boundary $w = b$, $n(b) = 0$. As w increases, $n(w)$ also increases. At the upper boundary $w = 1$, $n(1) = 0.05$. At the moment of starting the fund where $w = 1$, incentive fees contribute about one quarter of the manager’s total value $f(1) = 0.20$,

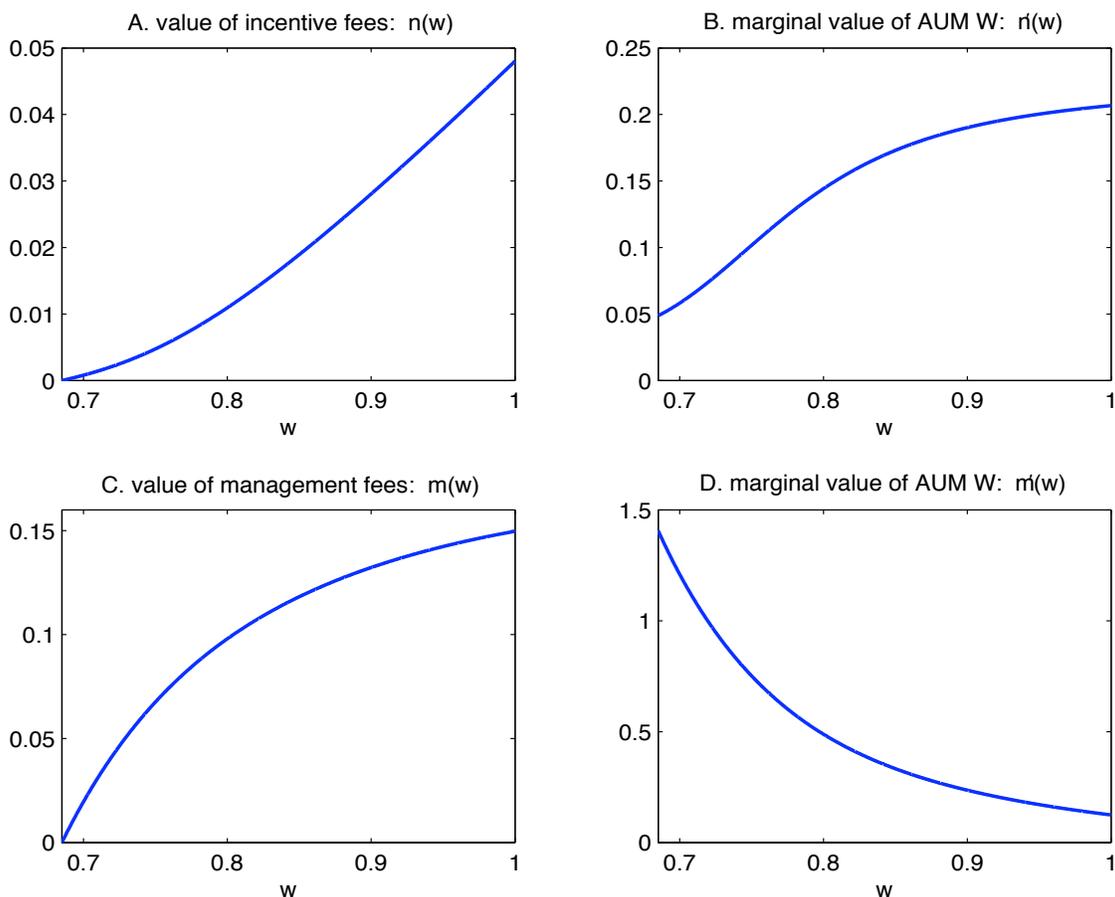


Figure 4: **The value of incentive fees $n(w)$, the value of management fees $m(w)$, and their sensitivities, $n'(w)$ and $m'(w)$.**

Panel B plots $n'(w)$, which is the “delta” for the value of incentive fees, using the option pricing terminology. Incentive fees are a sequence of embedded call options, and $N(W, H)$ is thus convex in the AUM W . At the liquidation boundary $b = 0.685$, the delta for incentive fees equals $n'(b) = 0.05$. As w increases, incentive fee delta $n'(w)$ also increases and reaches the value of $n'(1) = 0.21$ at $w = 1$. Note that $n(w)$ is the present value of possibly an infinite sequence of incentive options for the manager to collect 20% of profits should the

AUM exceed the HWM in the future.

The value of management fees $m(w)$. Panel C plots $m(w)$. When the fund is liquidated, $m(b) = 0$. As w increases, $m(w)$ increases. At $w = 1$, $m(1) = 0.15$, which is 75% of the manager's total value, $f(1) = 0.20$. Quantitatively, management fees contribute more than incentive fees to the manager's total compensation. For our calibration, $m(1)$ is three times the value of incentive fees $n(1) = 0.05$. Intuitively, the management fee acts as a wealth tax on the AUM provided that the fund is alive. The present value of a flow based on wealth tax at 2% can thus be significant. For the private equity industry, Metrick and Yasuda (2010) also find that management fees contribute to the majority of total managerial compensation.

Panel D plots $m'(w)$. At the liquidation boundary $w = b$, $m'(b) = 1.41$, which is about 29 times of $n'(b) = 0.05$. This is not surprising because the vast majority of the manager's value $f(w)$ derives from management fees when w is near liquidation. The manager loses all future management fees when the fund is liquidated. Intuitively, the manager effectively holds a short position in the liquidation (put) option. As w increases, $m'(w)$ decreases. At $w = 1$, $m'(1) = 0.13$, which is lower than $n'(1) = 0.21$. This is not surprising because the incentive fee is very close to being in the money at $w = 1$.

The manager collects management fees as long as the fund survives, but only receives incentive fees when the AUM exceeds the HWM. Incentive fees, as a sequence of call options on the AUM, encourage managerial risk taking. Management fees, as a fraction c of the fund's AUM, effectively give the manager an un-levered equity *cash flow* claim in the fund provided the fund is alive. However, upon liquidation, the manager receives nothing and moreover loses all future fees. Therefore, fund liquidation is quite costly for the manager and the manager optimally chooses a prudent use of leverage for survival so as to collect management fees. Quantitatively, in our model, management fees dominate incentive fees in the manager's total compensation.

For expositional simplicity, we have so far intentionally chosen a parsimonious baseline model. In the next four sections, we extend our model along four important dimensions: risk seeking incentive under leverage constraint, managerial ownership, new money flow, and managerial outside restart option.

6 Risk seeking with leverage constraint

In our preceding analysis, the risk-neutral manager behaves in an effectively risk-averse manner, i.e. $\psi(w) > 0$ because the manager is sufficiently concerned about performance-triggered liquidation (drawdown) risk. Indeed, the manager's effective risk aversion is necessary to ensure that the manager's value is finite without other constraints.

We now generalize our model so that the manager can potentially behave in a risk-seeking manner. This generalization is important since it is often believed that incentive fees encourage managerial risk seeking. To ensure that the manager's value $f(w)$ is finite, we impose the following leverage constraint, which is also often referred to as a margin requirement,

$$\pi(w) \leq \bar{\pi}, \quad (29)$$

where $\bar{\pi} > 1$. For assets with different liquidity and risk profiles, margin requirement $\bar{\pi}$ may differ. For example, individual stocks have lower margins than Treasury securities do. See Ang et al. (2011) for a summary of various margin requirements for different assets.

With the leverage constraint (29), the optimal investment strategy $\pi(w)$ is given by

$$\pi(w) = \min \left\{ \frac{\alpha}{\sigma^2 \psi(w)}, \bar{\pi} \right\}, \quad (30)$$

where $\psi(w)$ is given by (18). When the constraint (29) does not bind, the manager behaves in an effectively risk-averse manner, $\psi(w) > 0$. Let \bar{w} denote the minimal level of w such that the leverage constraint (29) binds, which implies $\pi(w) = \bar{\pi}$ for $\bar{w} \leq w \leq 1$. Because \bar{w} is optimally chosen by the manager, the manager's value $f(w)$ is twice continuously differentiable at $w = \bar{w}$, in that the following conditions are satisfied at \bar{w} ,

$$f(\bar{w}_-) = f(\bar{w}_+), \quad f'(\bar{w}_-) = f'(\bar{w}_+), \quad f''(\bar{w}_-) = f''(\bar{w}_+). \quad (31)$$

Dynamic leverage $\pi(w)$ and effective risk attitude $\psi(w)$. Figure 5 plots dynamic leverage policy $\pi(w)$ and the corresponding “effective” risk attitude $\psi(w)$ for three levels of the liquidation boundary $b = 0.2, 0.5$, and 0.685 . We set the margin constraint $\bar{\pi} = 4$ and keep remaining parameter values the same as those for the baseline calibration.

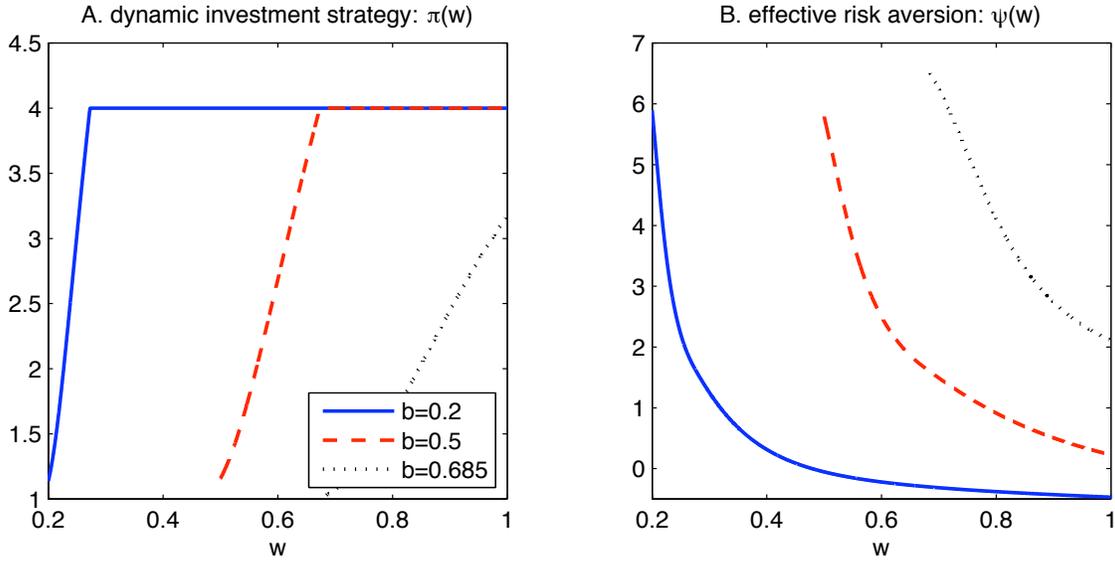


Figure 5: **Dynamic leverage $\pi(w)$ and managerial risk attitude $\psi(w)$.**

For the case with $b = 0.685$, the manager chooses prudent leverage and the margin constraint $\pi \leq 4$ never binds. As we decrease b from 0.685 to 0.5, the manager becomes less concerned about liquidation and hence increases leverage $\pi(w)$ causing the leverage constraint (29) to bind for $w \geq \bar{w} = 0.67$. However, the manager remains effectively risk averse, i.e. $\psi(w) > 0$.

Importantly, the manager may engage in risk seeking with low liquidation risk. Consider the case with $b = 0.2$. The margin requirement $\pi \leq 4$ binds for $w \geq \bar{w} = 0.27$ and thus in this region, the manager chooses the maximally allowed leverage $\bar{\pi} = 4$.⁵ In the region with sufficiently low w , i.e. when $0.27 \leq w \leq 0.48$, the manager behaves in a risk-averse manner ($\psi(w) < 0$). See the part of the solid line below the horizontal dashed line in Panel B. We note that even with risk-seeking behavior, the manager still behaves prudently for low w . This is opposite to the typical risk-seeking intuition in option-based models. We often hear that managers become risk seeking as the firm gets close to liquidation because the manager effectively holds an option with asymmetric payoff. However, this intuition does not apply here because debt in our model is risk-free and hence the typical argument that risk shifting is optimal when the firm does poorly does not work in our model. Again, survival concern

⁵In the risk-seeking region, margin requirement such as (29) is needed to ensure the convergence of the optimization problem.

is the dominant force in our model when the fund is performing poorly because there is no wealth transfer from the creditors to the fund and/or the manager.

7 Managerial ownership

Hedge fund managers often have equity positions in the fund that they run, which potentially mitigates managerial conflicts with investors. In this section, we analyze the effects of managerial ownership on leverage and valuation.

7.1 Model setup and solution

Let ϕ denote managerial ownership in the fund. For simplicity, we assume that ϕ remains constant over time. Let $Q(W, H)$ denote the manager's total value including both the value of total fees $F(W, H)$ and the manager's *pro rata* share of investors' value $\phi E(W, H)$,

$$Q(W, H) = F(W, H) + \phi E(W, H). \quad (32)$$

The manager dynamically chooses the investment policy π to maximize (32). Using the homogeneity property, we write $Q(W, H) = q(w)H$ where $q(w)$ is the manager's scaled total value. In Appendix, we show that the optimal investment strategy $\pi(w)$ is given by

$$\pi(w) = \frac{\alpha}{\sigma^2 \psi_q(w)}, \quad (33)$$

where $\psi_q(w)$ is the manager's effective risk aversion defined by

$$\psi_q(w) = -\frac{wq''(w)}{q'(w)}. \quad (34)$$

With managerial ownership ϕ , the manager's effective risk aversion $\psi_q(w)$ and hence leverage $\pi(w)$ naturally depend on $q(w)$, the sum of the value of fees $f(w)$ and the value of the fund's ownership $\phi e(w)$. The Appendix provides the ODE and boundary conditions for $q(w)$.

7.2 Results

We first illustrate how leverage $\pi(w)$ depends on the manager's equity ownership, and then analyze the quantitative implications of managerial ownership.

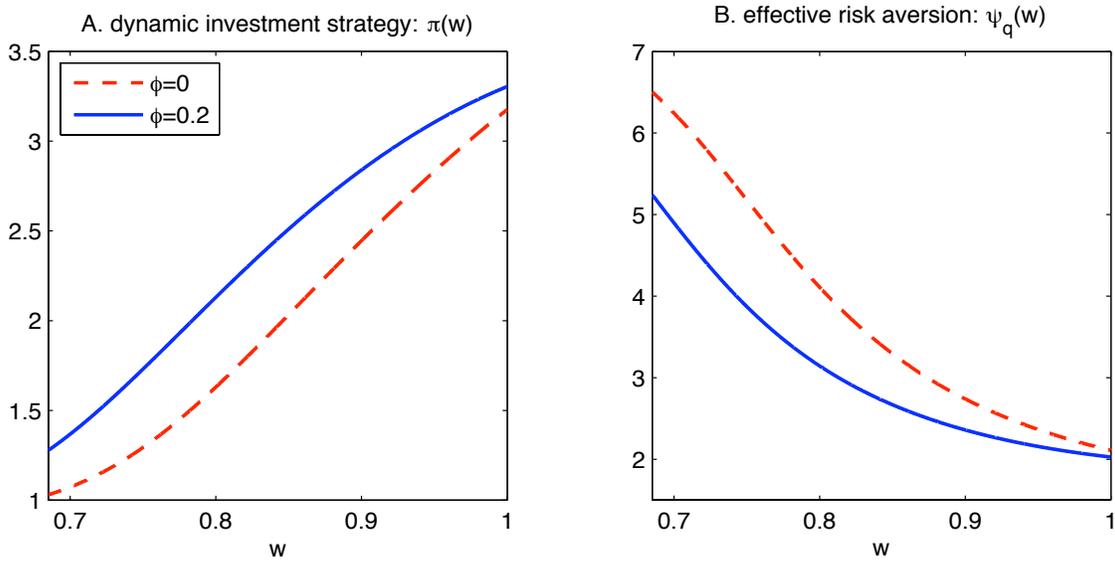


Figure 6: **Dynamic investment strategy** $\pi(w)$ and “**effective**” risk aversion $\psi_q(w)$ with managerial ownership ($\phi = 0.2$).

Dynamic leverage $\pi(w)$ and effective risk aversion $\psi_q(w)$. Figure 6 plots the manager’s optimal leverage $\pi(w)$ and the risk aversion measure $\psi_q(w)$ for $\phi = 0.2$ and $\phi = 0$ (the baseline case). All other parameter values are the same for the two cases. Leverage $\pi(w)$ is higher and correspondingly the effective risk aversion $\psi_q(w)$ is lower, with managerial ownership $\phi = 0.2$ than without ownership, $\phi = 0$, *ceteris paribus*. The larger the inside equity position ϕ , the more the manager cares about investors’ value $e(1)$, encouraging the manager to choose a higher leverage.

Quantitative results. Table 1 shows the effects of managerial ownership ϕ on leverage and values. As we increase ϕ from 0 to 20%, leverage $\pi(1)$ increases from 3.18 to 3.30, the value of management fees $m(1)$ decreases from 15% to 13.9%, the value of incentive fees $n(1)$ increases from 4.8% to 5.7%, and investors’ value $e(1)$ increases from 1 to 1.02. Inside equity ownership improves incentive alignments between the manager and investors by making the manager less concerned about liquidation risk/friction and encouraging the manager to reduce under-investment and choose a higher leverage.

Managerial ownership increases the value of incentive fees $n(1)$ but lowers the value of management fees $m(1)$, as leverage makes liquidation more likely. However, the quantitative

Table 1: THE EFFECTS OF MANAGERIAL OWNERSHIP.

The parameter values are: $r - g = 0$, $\delta + \lambda = 10\%$, $\alpha = 1.22\%$, $\sigma = 4.26\%$, $b = 0.685$, $c = 2\%$, and $k = 20\%$.

ϕ	$\pi(1)$	$m(1)$	$n(1)$	$f(1)$	$e(1)$	$v(1)$
0	3.1753	0.1497	0.0480	0.1977	1	1.1977
10%	3.2300	0.1442	0.0528	0.1970	1.0142	1.2112
20%	3.3038	0.1388	0.0568	0.1956	1.0242	1.2197
50%	3.5188	0.1255	0.0646	0.1901	1.0408	1.2309

effect on neither $n(1)$ nor $m(1)$ is significant. Moreover, because $m(1)$ and $n(1)$ move in opposite directions with comparable magnitudes, the manager's total value $f(1)$ remains about 20%, effectively unchanged. While ownership influences the manager's leverage choice and various values, the quantitative effects of ownership are not large (even when $\phi = 50\%$, as we see from the last row of Table 1).

8 New money flow

Money chases after good performances in hedge funds. We next incorporate this feature into our model. We show that the manager benefits significantly from the new money inflows.

8.1 Model setup

We model performance-triggered money inflows as follows: whenever the fund's AUM exceeds its HWM, new money flows into the fund. Because the HWM H grows deterministically at the rate of $(g - \delta)$ in the interior region $W < H$, the fund's AUM only exceeds its HWM and new money subsequently flows in, when $dH_t - (g - \delta)H_t dt > 0$.

Let dI_t denote the new money inflows over time increment $(t, t + \Delta t)$. We assume that dI_t is proportional to the performance measure $dH_t - (g - \delta)H_t dt > 0$, in that

$$dI_t = i [dH_t - (g - \delta)H_t dt] , \quad (35)$$

where the constant parameter $i > 0$ measures the sensitivity of dI_t with respect to the fund's performance. For example, suppose $W_t = H_t = 100$ and the next year's realized AUM is

$W_{t+1} = 115$. Then, the manager collects $3 = 20\% \times 15$ in incentive fees. With $i = 0.8$, the new money inflow is $dI_t = 12 = 0.8 \times 15$, which is about 12% of the fund's AUM.

Including the new money inflow term into (4), we write the AUM dynamics as,

$$\begin{aligned} dW_t = & \pi_t W_t (\mu dt + \sigma dB_t) + (1 - \pi_t) r W_t dt - \delta W_t dt - c W_t dt \\ & - (k - i) [dH_t - (g - \delta) H_t dt] - dJ_t, \quad t < \tau. \end{aligned} \quad (36)$$

There are two types of investors, current and future investors. Let $E(W, H)$ denote the PV of all investors including both current and future investors. Future investors have not contributed any capital yet since their contributions will be in the form of new (stochastic) money flows. Nonetheless, we can still value them. Let $E_1(W, H)$ and $E_2(W, H)$ denote the PV of investments for current and future investors, respectively. In equilibrium, we need to ensure that both current and future investors are willing to invest in the fund. The manager benefits from the new money inflow in two ways. New money inflow increases the fund's AUM and hence management fees. Additionally, should the AUM exceeds the HWM in the future, the manager will collect incentive fees on a larger asset base.

8.2 Model solution

Leverage formula and valuation formulas for fees. New money significantly influences the manager's value, investors' payoffs, and leverage π . Given $f(w)$, the optimal leverage $\pi(w)$ is still given by the mean-variance formula, (17)-(18). The Appendix characterizes the ODE and boundary conditions for $f(w)$, $m(w)$, and $n(w)$.

Current investors' value $e_1(w)$. In the appendix, we characterize the ODE and boundary conditions for $e_1(w)$. The current investors' voluntary participation condition is

$$e_1(1) \geq 1. \quad (37)$$

We next turn to the PV of (future) new money inflow from future investors. First, we calculate "the total discounted amount of all future money inflows." Then, we assess the PV of the total discounted amount of new money inflow.

The total expected discounted amount of new money flow. New money flows in when the fund's AUM exceeds its HWM. Let $X(W, H)$ denote the total expected discounted

amount of all future money inflows, which is given by

$$X(W, H) = \mathbb{E}_t \left[\int_t^\tau e^{-r(s-t)} i [dH_s - (g - \delta)H_s ds] \right], \quad (38)$$

where τ is stochastic liquidation time. Because new money can only possibly flow in at $w = 1$, current and future investors share the same HWM. This property is desirable because we only need to track a single fund-wide HWM for all investors. Consequently, this richer specification with new money flow is as tractable as the baseline model with a single investor.

We note that $X(W, H)$ is the discounted amount of future capital. In order to account for the manager's alpha-generating technology on the new capital contributed by future investors, we next calculate the PV of the future investors' contribution $E_2(W, H)$. Again, using the homogeneity, we characterize $e_2(w)$.

The PV of future investors' contributed capital $e_2(w)$. In the appendix, we characterize the ODE and boundary conditions for $e_2(w)$. The difference between the value of future investors $e_2(w)$ and the discounted amount of new money inflow $x(w)$ gives the net surplus for future investors created by the alpha strategy.

The fund's total net surplus $z(w)$. The scaled fund's value $v(w)$ equals the sum of all investors' value $e(w)$ and the manager's value $f(w)$. Summing the existing capital w and discounted amount of future money inflows $x(w)$ gives the total capital, $w + x(w)$. Let $z(w)$ denote the scaled total net surplus, which equals the difference between $v(w)$ and $w + x(w)$,

$$z(w) = v(w) - (w + x(w)) = e_1(w) + e_2(w) + f(w) - (w + x(w)). \quad (39)$$

8.3 Results

Dynamic investment strategy $\pi(w)$ and effective risk aversion $\psi(w)$. Figure 7 plots leverage $\pi(w)$ and the manager's effective risk aversion $\psi(w)$ for the case with $i = 0.8$ and compares with the baseline case where $i = 0$.

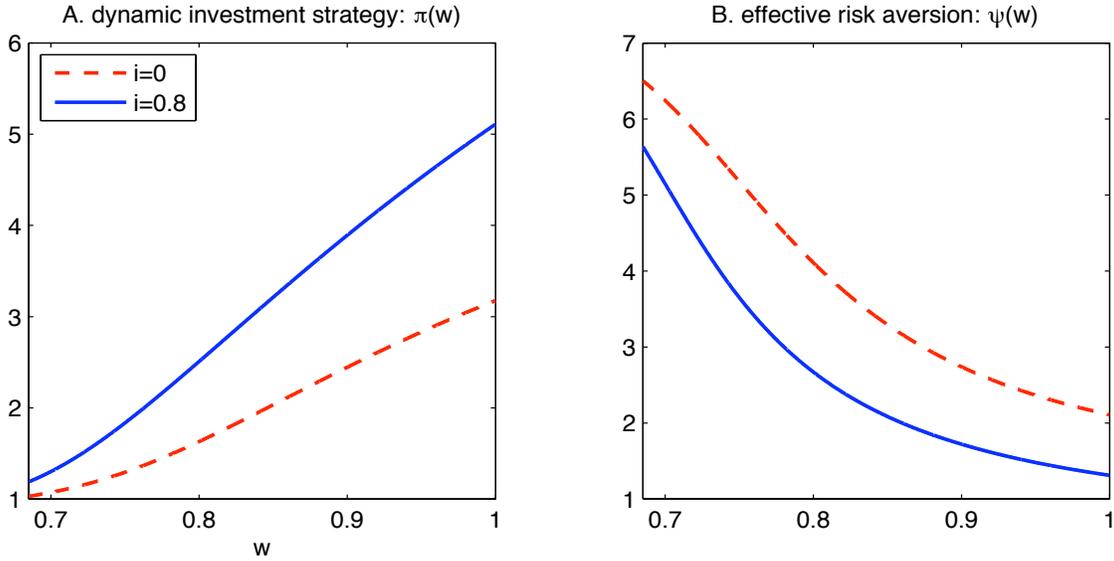


Figure 7: **Dynamic investment strategy $\pi(w)$ with new money inflow.**

The new money inflow rewards the manager by increasing the AUM upon which management and incentive fees are based in the future. The higher the new money inflow i , the less risk-aversely the manager behaves (lower $\psi(w)$) and the higher the leverage. Additionally, the higher the value of w , the more valuable the new money to the manager, the lower the manager's effective risk aversion $\psi(w)$, and the higher the leverage $\pi(w)$. At $w = 1$, when $i = 0.8$, leverage is $\pi(1) = 5.11$, which is 61% higher than the leverage, $\pi(1) = 3.18$ in the baseline case with no new money inflow; correspondingly, the manager's effective risk aversion is $\psi(1) = 1.31$, which is 38% lower than the baseline case with $i = 0$.

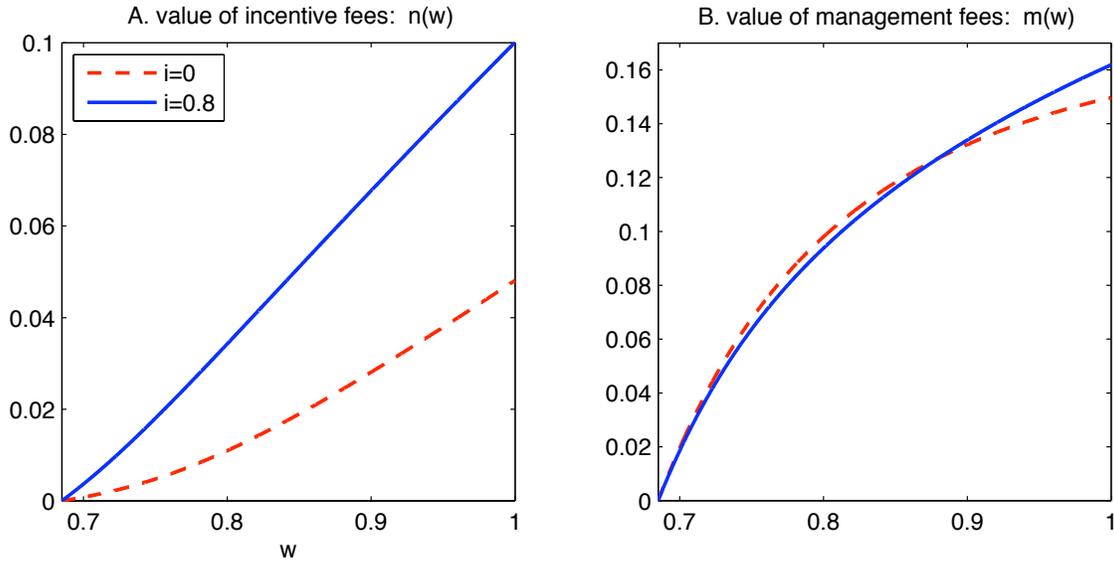


Figure 8: **PV of incentive fees $n(w)$ and PV of management fees $m(w)$.**

Figure 8 plots the value of incentive fees $n(w)$ and the value of management fees $m(w)$. This figure is consistent with our intuition that $n(w)$ responds strongly to the new money inflow sensitivity i , while management fees $m(w)$ responds much less to i . Panel A shows that the higher the value of w , the stronger the effect of new money inflow i on $n(w)$. Interestingly, new money inflow has two opposing effects on $m(w)$. On one hand, the new money inflow increases future AUMs, which in turn increase $m(w)$. On the other hand, future new money flow encourages leverage and hence increases the liquidation risk, which in turn lowers $m(w)$. As a result, for sufficiently high w , $m(w)$ increases with the new money inflow i because the positive AUM effect dominates the negative liquidation effect. For sufficiently low w , the opposite holds. Therefore, new money flow has ambiguous effects on $m(w)$ as shown in Panel B. Quantitatively, the effect of new money inflow is not strong.

Comparative statics with respect to new money flow i . In Table 2, we turn to the comparative static results of the new money inflow parameter i . We keep all other parameter values the same as in the baseline case with $i = 0$.

The quantitative effect of new money flow is significant. As we increase i from 0 to 1, leverage $\pi(1)$ increases significantly from 3.18 to 5.62, and the manager's total value $f(1)$ increases by 45% from 0.2 to 0.29. Mostly the new money flow effect operates through the

Table 2: THE EFFECTS OF NEW MONEY FLOW

The parameter values are: $r - g = 0$, $\delta + \lambda = 10\%$, $\alpha = 1.22\%$, $\sigma = 4.26\%$, $b = 0.685$, $c = 2\%$, and $k = 20\%$.

i	$\pi(1)$	$m(1)$	$n(1)$	$f(1)$	$e_1(1)$	$x(1)$	$e_2(1)$	$z(1)$
0	3.1753	0.1497	0.0480	0.1977	1	0	0	0.1977
0.2	3.6565	0.1511	0.0574	0.2085	1.0107	0.0573	0.0580	0.2199
0.5	4.3743	0.1550	0.0750	0.2300	1.0247	0.1873	0.1920	0.2594
0.8	5.1086	0.1618	0.1001	0.2619	1.0367	0.4001	0.4148	0.3133
1.0	5.6197	0.1688	0.1240	0.2928	1.0429	0.6197	0.6463	0.3623

value of incentive fees $n(1)$, which increases by 2.6 times from 4.8% to 12.4%. New money flow rewards the manager when the fund is doing well and thus particularly influences the value of incentive fees $n(1)$. The value of management fees $m(1)$ also increases with i , because new money inflow causes a higher asset base so that the manager collects more fees in the future. Quantitatively, the new money flow has a much bigger effect on $n(1)$ than on $m(1)$.

With new money flow $i = 1$, out of the manager's total value $f(1) = 0.293$, management fees $m(1)$ account for about 58% and incentive fees contribute to the remaining 42%. With a smaller new money inflow $i = 0.5$, out of $f(1) = 0.23$, management fees and incentive fees account for about two thirds and one third, respectively. Therefore, even with new money, management fees continue to account for the majority of the manager's value.

While most benefits of the new money flow accrue to the manager, investors are also better off. The current investors' value $e_1(1)$ increases by 4% from 1 to 1.043 as the new money flow i increases from 0 to 1. Interestingly, future investors are also better off by 4% per unit of AUM. This is due to the property that all investors, current and future, in our model share the same HWM, which substantially simplifies our analysis. With $i = 1$, the total net surplus equals $z(1) = 36.2\%$, out of which the manager, current investors, and future investors collect 29.3%, 4.3%, and 2.6%, respectively.

We have thus far assumed that the manager loses everything and receives no outside option when the fund is liquidated. This assumption is unrealistically strong. In reality, hedge fund managers re-start new funds after fund liquidation or closure.⁶ We next generalize

⁶Using numerical solution in a discrete-time finite-horizon setting, Hodder and Jackwerth (2007) emphasize the option value of endogenous fund closure but not the option value of restarting a fund.

our model to allow the manager to have fund closure and restart options.

9 Restart options

We generalize the baseline model to incorporate manager’s restart options and then use the model to illustrate the quantitative implications of restart options on leverage and valuation.

9.1 Model setup and solution

The manager can voluntarily close the fund and start a new one whose size depends on the manager’s track record. We analyze a stationary framework with infinite restart options. In the appendix, we analyze the case with one restart option. The reality is likely to lie between the stationary and one-restart-option settings. In the end of this section, we provide sensitivity analysis with respect to the number of restart options.

A stationary model with infinite restart options. At any moment when the current fund’s AUM is W and its HWM is H , the manager has an option to start a new fund with a new AUM, which we denote as $S(W, H)$. Let ν denote the ratio between the new fund’s AUM $S(W, H)$ and the previous fund’s closing AUM W , i.e. $\nu = S(W, H)/W$. To illustrate the effects of restart options, we assume that the ratio ν satisfies

$$\nu(w) = \theta_0 + \theta_1 w + \frac{\theta_2}{2} w^2, \quad (40)$$

where $\theta_1 > 0$ and $\theta_2 < 0$. Intuitively, the better the fund’s performance, the larger ν . Additionally, the impact of w on ν diminishes as we increase w . The manager faces the following tradeoff with regard to the restart option. By closing the existing fund and starting a new one, the manager benefits by resetting the fund’s HWM and hence being much closer to collecting incentive fees but forgoes the fees on the closed fund. Additionally, the new fund’s AUM $S(W, H)$ may be smaller than the closed fund’s AUM W and it is costly to close the existing and start a new fund. The manager chooses the timing which influences the start-up AUM size of the new fund.

In the interior region ($w < 1$), we have an ODE similar to the one for the baseline model. Importantly, the new economics appears at the restart option boundary. Let $f_\infty(w)$ denote the manager’s scaled value function with infinite rounds of restart options. Let \bar{w}_∞ denote

the optimal threshold for the restart option. The manager chooses the optimal boundary \bar{w}_∞ so that the following value-matching and smooth-pasting conditions are satisfied,

$$f_\infty(\bar{w}_\infty) = \bar{w}_\infty \nu(\bar{w}_\infty) f_\infty(1), \quad (41)$$

$$f'_\infty(\bar{w}_\infty) = (\nu(\bar{w}_\infty) + \bar{w}_\infty \nu'(\bar{w}_\infty)) f_\infty(1). \quad (42)$$

Condition (41) requires that the manager's value $f_\infty(w)$ is continuous at the moment of abandoning the existing fund and starting a new fund. At the optimally chosen restart boundary \bar{w}_∞ given by the smooth pasting condition (42), the manager is just indifferent between starting the new fund or not. For $w < \bar{w}_\infty$, the manager immediately closes the existing fund and starts a new one.

An alternative and technically equivalent interpretation of our framework with restart options is that the manager has an option to reset the HWM following poor fund performance as the optionality embedded in incentive fees becomes significantly out of money. Resetting the HWM causes some investors to withdraw their capital and leave the fund. Both fund restart and HWM reset interpretations are consistent with our model.

9.2 Model results

Parameter choice and calibration. We calibrate the three new parameters, θ_0, θ_1 , and θ_2 , in (40) which determines the new fund's size, as follows. We target (1) the restart boundary W to be 80% of the fund's AUM H , i.e. $\bar{w}_\infty = 0.8$; (2) the subsequent fund's AUM to be 75% of the previous fund's AUM, i.e. $\nu(\bar{w}_\infty) = 0.75$; (3) the new fund's size is zero when the manager is forced to liquidate at $w = b$, i.e. $\nu(b) = 0$. Using these three conditions, we obtain $\theta_0 = -24.75$, $\theta_1 = 61.47$, and $\theta_2 = -74$. The AUM for each consecutive fund decreases by 25% from the previous fund's HWM in our calibration. Quantitatively, we show that only the first several restart options matter (due to discounting and shrinking fund sizes in the future following poor performances). For the comparison purpose, we use the same parameter values as in the baseline when feasible.

Dynamic investment strategy $\pi(w)$ and effective risk aversion $\psi(w)$. Figure 9 plots the optimal investment strategy $\pi(w)$ and the manager's effective risk aversion $\psi(w)$ for two cases, infinite restart options and no restart option as in the baseline case. Quantitatively,

the manager's restart options significantly increase leverage. For example, $\pi(1) = 4.08$ with ∞ restart options while $\pi(1) = 3.18$ in our baseline with no restart options. Correspondingly, the manager's effective risk aversion $\psi(1)$ falls from 2.11 to 1.64 at $w = 1$ due to restart options. At the moment of starting up the new fund, $\bar{w}_\infty = 0.8$ and leverage $\pi(0.8) = 2.20$, which is much larger than leverage $\pi(0.8) = 1.63$ in the baseline case with no restart option. Intuitively, the manager becomes more aggressive in deploying leverage because the manager is effectively less risk averse with restart options than without.

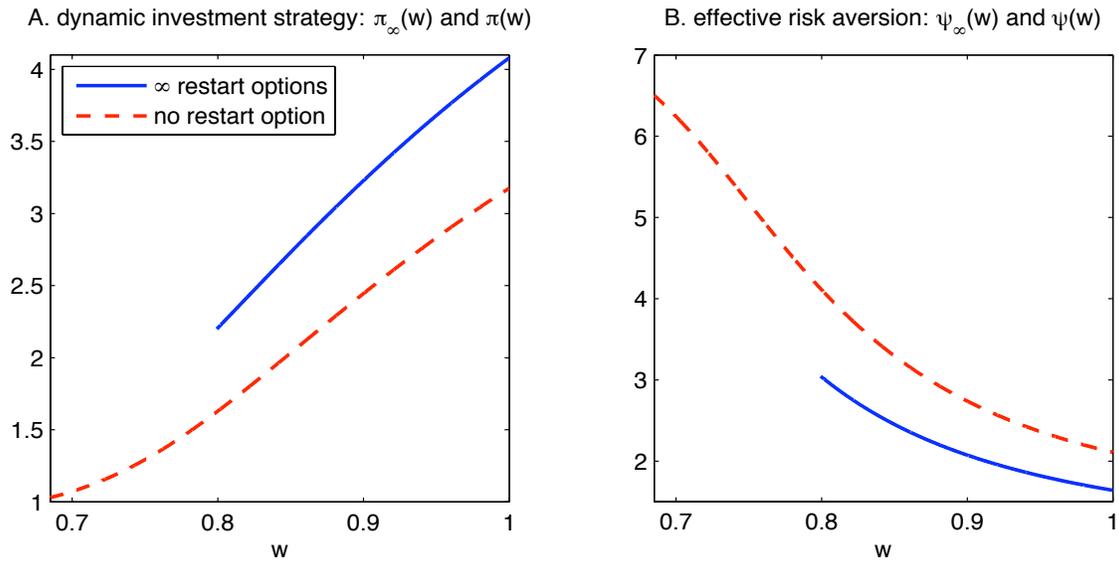


Figure 9: Investment strategy $\pi(w)$ and risk attitude $\psi(w)$ with restart options.

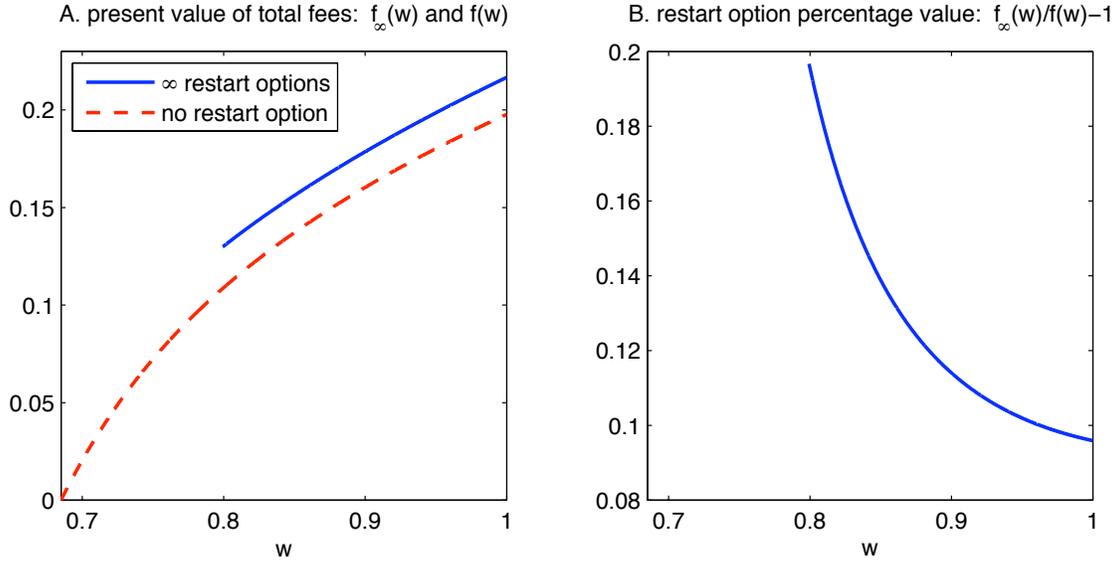


Figure 10: **PV of total fees $f(w)$ and relative value of restart options $f_1(w)/f_2(w)-1$.**

The PV of total fees $f(w)$ and the value of restart options. Figure 10 quantifies the value of restart options. At the optimally chosen restart option boundary $\bar{w}_\infty = 0.8$, $f_\infty(0.8) = 0.130$, which is about 20% higher than $f(0.8) = 0.109$ for the baseline case with no restart options. Even at $w = 1$ when restart option becomes least valuable, $f_\infty(1) = 0.217$, which is 10% higher than $f(1) = 0.198$ in the baseline. In summary, the value of restart options is large.

Present value of incentive fees and management fees. We now analyze the effects of restart options on the values of incentive fees and of management fees. Panel A of Figure 11 plots the value of incentive fees $n_\infty(w)$ with infinite restart options. The effect of restart options on incentive fees $n(w)$ is very large. At $w = 1$, $n_\infty(1) = 0.104$, which is 2.17 times of $n(1) = 0.048$ in the baseline case. At the optimal restart boundary $\bar{w}_\infty = 0.8$, $n_\infty(0.8) = 0.063$, which is 5.76 times of $n(0.8) = 0.011$ in the baseline case.

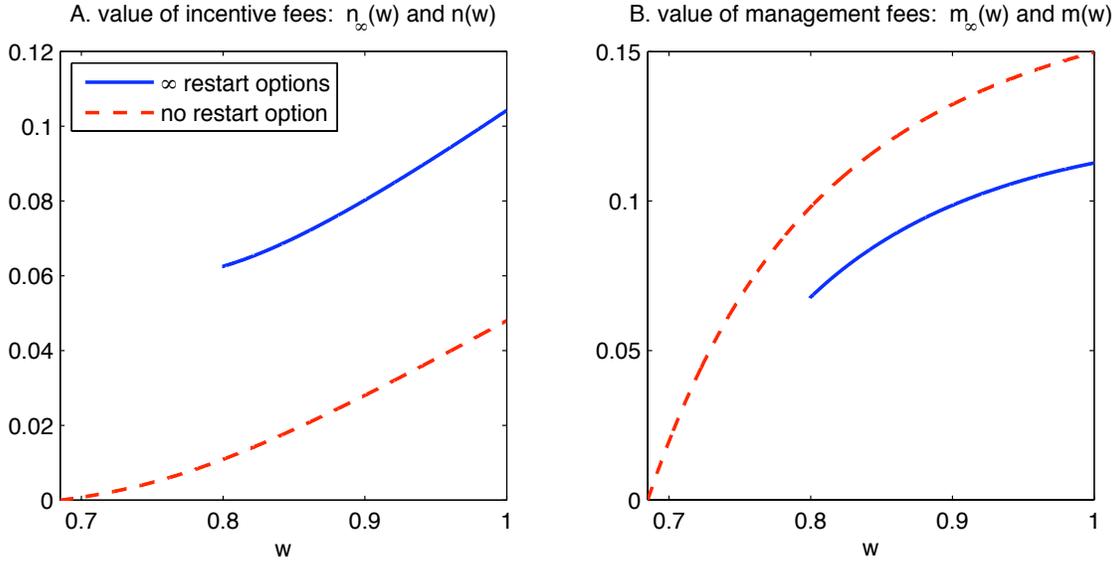


Figure 11: **Present value of incentive fees $n(w)$ and management fees $m(w)$.**

In contrast, restart options have *negative* effects on the value of management fees $m(w)$. At $w = 1$, $m_\infty(1) = 0.112$, which is 75% of the value of management fees $m(1) = 0.150$ in the baseline case. At the restart boundary $\bar{w}_\infty = 0.8$, $m_\infty(0.8) = 0.068$, which is 69% of the value of management fees $n(0.8) = 0.098$ in the baseline case. This seemingly counter-intuitive negative effect can be understood as follows. Restart options have two opposing effects on the value of management fees. First, restart options make the manager more aggressive with leveraging and abandoning the fund, which cause each fund to be shorter lived than the single fund in the baseline. With restart options, the manager collects management fees from many rounds of funds. Theoretically, the net effect of restart options on $m(w)$ can go either way. For our calibration, the negative effect of restart options on the current fund's management fees outweighs the positive effect of more funds causing $m(w)$ to be lower with the introduction of restart options.

The value of first, second, and remaining restart options. Having focused on the stationary case with infinite restart options, we now turn to the sensitivity analysis with respect to the number of restart options. We consider four cases where the manager has

Table 3: THE EFFECTS OF INCREASING THE NUMBER OF RESTART OPTIONS.

This table reports the first fund’s restart boundary \bar{w} , leverage $\pi(1)$, and various value functions as we increase the total number of the manager’s restart options.

Option #	\bar{w}	$\pi(1)$	$\psi(1)$	$m(1)$	$n(1)$	$f(1)$	$e(1)$	$v(1)$
0	0.685	3.1753	2.1105	0.1497	0.0480	0.1977	1	1.1977
1	0.788	3.6051	1.8587	0.1390	0.0675	0.2065	1.0070	1.2135
2	0.794	3.8204	1.7539	0.1308	0.0802	0.2110	1.0072	1.2182
∞	0.800	4.0791	1.6427	0.1124	0.1042	0.2166	1.0067	1.2233

zero, one, two, and infinite restart options.⁷ Table 3 reports the results in the first fund that the manager runs across the four cases. As we increase the number of restart options, the manager values future more, hence exits the current fund sooner, chooses a more aggressive investment strategy, and values incentive fees more. Surprisingly, with more funds to manage, the value of management fees $m(1)$ may still decrease with the number of restart options, which is seen in Table 3. Intuitively, more aggressive investment and exit strategies make the manager lose more management fees from the current fund than being potentially compensated from future funds’ management fees in PV.

Quantitatively, restart options have much stronger effects on the value of incentive fees than on the value of management fees. For our calculation, incentive fees increase from less than 5% in the baseline with no restart option to 10.4% with infinite restart options. Out of the total increase of the manager’s value $f(1)$, which is about 1.9%, from 19.8% in the baseline case to 21.7% in the stationary case, an increase of 1.3%, which is about two thirds of the total increase of $f(1)$, is attributed to the first two restart options. Our calibrated exercise thus suggests that the first few restart options carry most values for the manager.

10 Conclusions

Hedge fund managers are paid via management fees and incentive fees. For example, “two twenty” which refers to 2% management fees on the AUM, and 20% incentive fees on the profits, is a commonly used fund manager’s compensation contract. We develop a valua-

⁷See the appendix for the case with one restart option. For cases with multiple restart options, we have more complicated notations, but the analysis is essentially the same and is available upon request.

tion model where the manager dynamically chooses leverage to maximize the PV of both management and incentive fees from the current and future managed funds (with money inflow/outflow, fund closure/restart options, and investors liquidation options). Outside investors in each fund rationally participate in the fund given their beliefs about the managerial skills and leverage strategies. The manager trades off the value creation by leveraging on the alpha strategy and the cost of inefficient liquidation and redemption/drawdown.

In our model, the key state variable, denoted as w , is the ratio between the fund's AUM and its HWM. The risk-neutral manager has incentives to preserve the fund's going-concern value so as to collect fees in the future. This survival/precautionary motive causes the manager to behave in an effectively risk-averse manner. The greater liquidation risk and/or costs, the more prudently the manager behaves. Optimal leverage increases with alpha and decreases with variance. Additionally, leverage decreases with the manager's endogenously determined effective risk aversion, both of which change with w . The higher the value of w , the less likely the fund is liquidated, the more likely the manager collects the incentive fees, the less risk aversely the manager behaves, and the higher the leverage.

We further incorporate additional important institutional features into our framework. First, we show that the manager engages in risk seeking when liquidation risk is low. Under such a scenario, margin requirement or leverage constraint may be necessary to ensure that leverage and the manager's value are finite and economically sensible. Second, managerial ownership in the fund helps mitigate agency frictions. Third, we incorporate money flow-performance relation into our model and show that this relation has significant implications on the manager's value and leverage. Finally, we introduce the manager's options to start up new funds and find that these options are valuable.

Quantitatively, our calibration suggests that the manager needs to create significant value to justify their compensation contracts. Both management fees and incentive fees are important contributors to the manager's value. Our baseline calibration suggests that the manager needs to create 20% value surplus on the AUM to justify their two-twenty contracts. Out of the manager's total value creation of 20 cents on a dollar, 75% is attributed to management fees (15 cents) and the remaining 25% is due to incentive fees (5 cents). By incorporating features such as new money flow, fund restart (HWM reset) options, and managerial ownership, we find that incentive fees contribute much more to the manager's

value. However, it seems robust that management fees carry a significant fraction, 50% or more of the total manager's value.

In reality, managerial skills may be unknown and time-varying. Learning about unknown managerial skills is a topic for future research. Moreover, managers with no skills may pretend to be skilled in order to collect fees. It is thus important for investors to infer and learn about managerial skills. While we have developed a single fund manager's leverage policy, we plan to integrate our model into an industry equilibrium setting where managers have different skills/ α .

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Table 4: SUMMARY OF KEY VARIABLES AND PARAMETERS

This table summarizes the symbols for the key variables used in the model and the parameter values.

Variable	Symbol	Parameters	Symbol	Value
Risk-free asset	D	Risk-free rate	r	5%
Assets under management	W	Un-levered alpha	α	1.22%
Cumulative returns	R	Volatility	σ	4.26%
High-water mark	H	Growth rate of H	g	5%
Leverage	π	Investors' withdrawal	δ	0
Outside liquidation risk	J	Probability of liquidation	λ	10%
Present value of management fee	M	Management fee	c	2%
Present value of incentive fee	N	Incentive fee	k	20%
Present value of investors' payoff	E	Lower liquidation boundary	b	0.685
Present value of total fee	F	Ownership of equity	ϕ	0.2
Total value of the fund	V	New money inflow rate	i	0.8
manager's total value with ownership	Q	Restart option parameter	θ_0	-24.75
Effective risk aversion	ψ	Restart option parameter	θ_1	61.47
Point of leverage constraint binding	\bar{W}	Restart option parameter	θ_2	-74
New fund's size	S	Margin requirement	$\bar{\pi}$	4

Appendix

A Technical details

For Section 3. We conjecture that the value function $F(W, H)$ takes the following homogeneous form in W and H :

$$F(W, H) = f(w)H, \quad (\text{A.1})$$

where $w = W/h$. When the manager's value is given by (A.1), we have

$$F_W(W, H) = f'(w), \quad F_{WW}(W, H) = f''(w)/H, \quad F_H(W, H) = f(w) - wf'(w). \quad (\text{A.2})$$

Substituting them into the HJB equation (12) and the boundary conditions (15)-(16), we obtain ODE (20) with boundary conditions (21)-(22). Substituting (A.1) and (A.2) into the leverage FOC (13), we obtain the optimal leverage formula given in (17). Applying the Ito's formula to (2) and (4), we obtain the (optimally controlled) stochastic process (19) for w .

For Section 5. Applying the standard differential equation pricing method to $M(W, H)$ defined in (5) and $N(W, H)$ defined in (6), we obtain the ODE (24) and (25) for $m(w)$ and $n(w)$, respectively. For the boundary behavior at $W_t = H_t$, we use the same argument as the one for $F(W, H)$. Consider the scenario where the asset value increases by ΔH over a small time interval Δt , the HWM is then re-set to $H + \Delta H$. The continuity of value functions before and after the adjustment of the HWM imply

$$M(H + \Delta H, H) = M(H + \Delta H - k\Delta H, H + \Delta H), \quad (\text{A.3})$$

$$N(H + \Delta H, H) = k\Delta H + N(H + \Delta H - k\Delta H, H + \Delta H). \quad (\text{A.4})$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$kM_W = M_H, \quad kN_W = k + N_H. \quad (\text{A.5})$$

Using the homogeneity property, we obtain the boundary conditions (27) and (28) for $m(w)$ and $n(w)$ at $w = 1$, respectively. At the lower liquidation boundary ($W = bH$), the manager loses both of management fees and incentive fees, and hence (26) holds.

For Section 7. With managerial ownership, in the region $W < H$, the manager's total value $Q(W, H)$ solves

$$(r + \lambda)Q(W, H) = \max_{\pi} [c + \phi(\delta + \lambda)]W + [\pi\alpha + (r - \delta - c)]WQ_W(W, H) + \frac{1}{2}\pi^2\sigma^2W^2Q_{WW}(W, H) + (g - \delta)HQ_H(W, H). \quad (\text{A.6})$$

The FOC for leverage π is

$$\alpha WQ_W(W, H) + \pi\sigma^2W^2Q_{WW}(W, H) = 0. \quad (\text{A.7})$$

Using the homogeneity property, $Q(W, H) = q(w)H$, we simplify (A.6) and obtain the following ODE for $q(w)$:

$$(r - g + \delta + \lambda)q(w) = [c + \phi(\delta + \lambda)]w + [\pi(w)\alpha + r - g - c]wq'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2q''(w). \quad (\text{A.8})$$

The optimal leverage $\pi(w)$ is given by (33)-(34). Because $q(w) = f(w) + \phi e(w)$, the lower boundary condition becomes $q(b) = \phi b$. Using the same analysis as the one for $F(W, H)$, we obtain the upper boundary condition $q(1) = (k + 1)q'(1) - k$.

For Section 8. We provide the ODEs and boundary conditions for various value functions in three steps: (1) $f(w)$, $m(w)$, and $n(w)$; (2) the current investors' value $e_1(w)$; (3) the total expected discounted amount of new capital $x(w)$; and (4) the PV of future investors' contributed capital $e_2(w)$.

(1) The PVs of total fees, management, and incentive fees: $f(w)$, $m(w)$, and $n(w)$. The continuity of the manager's value before and after hitting the HWM implies $F(H + \Delta H, H) = k\Delta H + F(H + \Delta H - k\Delta H + i\Delta H, H + \Delta H + i\Delta H)$. By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$(k - i)F_W(H, H) = k + (1 + i)F_H(H, H). \quad (\text{A.9})$$

Using the homogeneity property, we simplify the boundary condition (A.9) as

$$f(1) = \frac{(k + 1)f'(1) - k}{1 + i}. \quad (\text{A.10})$$

Similarly, we may also obtain the following boundary conditions for $m(w)$ and $n(w)$:

$$m(1) = \frac{(k+1)m'(1)}{1+i}, \quad (\text{A.11})$$

$$n(1) = \frac{(k+1)n'(1) - k}{1+i}. \quad (\text{A.12})$$

At the liquidation boundary $w = b$, the manager collects no fees, which implies (A.13),

$$f(b) = m(b) = n(b) = 0. \quad (\text{A.13})$$

Now we turn to analyze the PV of current investor's payoff $E_1(W, H)$.

(2) Current investors' value $e_1(w)$. First, we turn to the boundary $W = H$. The continuity of value function implies

$$E_1(H + \Delta H, H) = -i\Delta H E_1(H + \Delta H, H) + E_1(H + \Delta H(1 - k + i), H + \Delta H + i\Delta H). \quad (\text{A.14})$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$(k - i) \frac{\partial E_1(H, H)}{W} = -iE_1(H, H) + (1 + i) \frac{\partial E_1(H, H)}{H}. \quad (\text{A.15})$$

Simplifying the above condition yields

$$e_1(1) = (k + 1)e_1'(1). \quad (\text{A.16})$$

Using the standard pricing method, the current investors' value $e_1(w)$ solves

$$(r - g + \delta + \lambda)e_1(w) = (\delta + \lambda)w + [\pi(w)\alpha + r - g - c]we_1'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2e_1''(w), \quad (\text{A.17})$$

with the boundary conditions,

$$e_1(b) = b, \quad (\text{A.18})$$

$$e_1(1) = (k + 1)e_1'(1), \quad (\text{A.19})$$

(3) The total expected discounted amount of new money flow $x(w)$. At the moment when new money flows in, $X(W, H)$ satisfies the value matching condition,

$$X(H + \Delta H, H) = i\Delta H + X(H + \Delta H - k\Delta H + i\Delta H, H + \Delta H + i\Delta H). \quad (\text{A.20})$$

Here, the first term on the right side of (A.20) is the amount of new money inflow and the second term gives the value of X after incentive fee payment and new money inflow. By taking the limit $\Delta H \rightarrow 0$ and using Taylor's expansion rule, we obtain

$$(k - i)X_W(H, H) = i + (1 + i)X_H(H, H). \quad (\text{A.21})$$

Using the homogeneity property, we simplify (A.21) as follows,

$$x(1) = \frac{(k + 1)x'(1) - i}{1 + i}. \quad (\text{A.22})$$

The homogeneity property also implies that $x(w)$ satisfies the following ODE,

$$(r - g + \delta + \lambda)x(w) = [\pi(w)\alpha + r - g - c]wx'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2x''(w), \quad b < w < 1. \quad (\text{A.23})$$

From ODE (A.23), there is no money inflow when $w < 1$. In sum, $x(w)$ solves (A.23) subject to boundary condition (A.22) and $x(b) = 0$, the condition at the liquidation boundary b .

(4) The PV of future investors' contributed capital $e_2(w)$. At the upper boundary $W = H$, the continuity of value function implies

$$E_2(H + \Delta H, H) = i\Delta H E_1(H + \Delta H, H) + E_2(H + \Delta H - k\Delta H + i\Delta H, H + \Delta H + i\Delta H). \quad (\text{A.24})$$

By taking the limit as ΔH approaches zero and using Taylor's expansion rule, we obtain

$$(k - i)\frac{\partial E_2(H, H)}{W} = iE_1(H, H) + (1 + i)\frac{\partial E_2(H, H)}{H}. \quad (\text{A.25})$$

Simplifying the above condition yields

$$e_2(1) = \frac{(k + 1)e_2'(1) - ie_1(1)}{1 + i}. \quad (\text{A.26})$$

The future investors' scaled value $e_2(w)$ satisfies the following ODE

$$(r - g + \delta + \lambda)e_2(w) = [\pi(w)\alpha + r - g - c]we_2'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2e_2''(w), \quad (\text{A.27})$$

with the boundary conditions,

$$e_2(b) = 0, \quad (\text{A.28})$$

$$e_2(1) = \frac{(k + 1)e_2'(1) - ie_1(1)}{1 + i}. \quad (\text{A.29})$$

For Section 9. In this appendix, we characterize the solution when the manager has two start-up options. That is, should the manager close the first fund whose AUM size and HWM at the moment of closure are denoted as W^1 and H^1 , respectively, the manager has an option to start up a new fund whose size is denoted as W^2 , equals $S(W^1, H^1) = \nu(w^1)W^1$, where $w^1 = W^1/H^1$ and the function $\nu(w)$ is given in (40).

At the moment of closing the current fund and starting a new one, value is continuous,

$$F^1(W^1, H^1) = F^2(\nu(w)W^1, \nu(w)W^1). \quad (\text{A.30})$$

Note that the HWM is re-set when the manager restarts the fund (see the right side of (A.30)). Additionally, the optimal control implies the smooth-pasting condition. Let \bar{w}_1 be the optimal boundary to restart the new fund. We thus have

$$f_1(\bar{w}_1) = \bar{w}_1 \nu(\bar{w}_1) f_2(1), \quad (\text{A.31})$$

$$f_1'(\bar{w}_1) = (\nu(\bar{w}_1) + \bar{w}_1 \nu'(\bar{w}_1)) f_2(1), \quad (\text{A.32})$$

where (A.32) captures the manager's optimal exercising of the exit/restart option.