## Calculating Greeks in Monte Carlo

Craig Pirrong Bauer College of Business University of Houston

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We've seen that the Greeks are important to market participants. We've also seen that two of the most important Greeks–Delta and Gamma–fall right out of finite difference algorithms. Things aren't quite so nice with Monte Carlo.

One way to estimate Delta is merely to estimate the option value using MC with two slightly different initial prices, e.g.,  $S_0 = 50$  and  $S_0 = 50.01$ , taking the difference between the estimates, and then dividing through by the assumed price difference (.01 in this example). (Use the same set of random numbers in each simulation.) This tends to be relatively inaccurate, however, because it blows up errors (because you are dividing errors by a small number).

There are two better alternatives. The first is path-wise estimation. Note that for a non-path dependent derivative, MC estimates

$$V(S_0) = \tilde{E}e^{-r(T-t)}f(S_T)$$

so

$$\Delta = \frac{\partial V}{\partial S_0} = \tilde{E} e^{-r(T-t)} \frac{\partial f}{\partial S_T} \frac{\partial S_T}{\partial S_0}$$

Consider a European call. Here  $f(S_T) = \max[S_T - K, 0]$ , so

$$\frac{\partial f}{\partial S_T} = e^{-r(T-t)} \mathbf{1}_{\mathbf{S}_T \ge \mathbf{K}}$$

Recall:

$$S_T = S_0 e^{(r-.5\sigma^2)(T-t) + \sigma\sqrt{T-t}Z}$$

where Z is our normal variate. Thus,

$$\frac{\partial S_T}{\partial S_0} = e^{(r-.5\sigma^2)(T-t) + \sigma\sqrt{T-t}Z} = \frac{S_T}{S_0}$$

Thus, when running MC simulations, to calculate  $\Delta$  also calculate:

$$\tilde{\Delta} = \frac{1}{N} e^{-r(T-t)} \sum_{i=1}^{N} \mathbf{1}_{\mathbf{S}_{T}(\mathbf{Z}_{i}) \ge \mathbf{K}} \frac{S_{T}(Z_{i})}{S_{0}}$$

This method can also be applied to path dependent options. Consider an Asian option with a payoff dependent on the average price  $\bar{S} = \sum_{j=1}^{m} S(t_j)$ . Here:

$$\frac{\partial f}{\partial S_T} = e^{-r(T-t)} \mathbf{1}_{\bar{\mathbf{S}} \ge \mathbf{K}}$$

Further:

$$\frac{\partial S(t_j)}{\partial S_0} = \frac{S(t_j)}{S_0}$$

implying:

$$\frac{\partial V_{Asian}}{\partial S_0} = e^{-r(T-t)} \mathbf{1}_{\bar{\mathbf{S}} \ge \mathbf{K}} \sum_{j=1}^m \frac{S(t_j)}{S_0}$$

Thus:

$$\tilde{\Delta}_{Asian} = \frac{1}{N} e^{-r(T-t)} \sum_{i=1}^{N} \mathbf{1}_{\bar{\mathbf{S}}_i \ge \mathbf{K}} \sum_{j=1}^{m} \frac{S_i(t_j)}{S_0}$$

where  $S_i(t_j)$  is the simulated price at  $t_j$  on simulation run *i*.

The second method is the Likelihood Ratio Method. Note that for a non-path dependent claim:

$$V(S_0) = e^{-r(T-t)} \int_0^\infty f(S_T) \frac{e^{-.5\psi^2(S_T)}}{S_T \sigma \sqrt{2\Pi(T-t)}} dS_T$$

where:

$$\psi(S_T) = \frac{\ln(\frac{S_T}{S_0}) - (r - .5\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}}$$

Denote:

$$g(S_T) = \frac{e^{-.5\psi^2(S_T)}}{S_T \sigma \sqrt{2\Pi(T-t)}}$$

and

$$g'(S_T) = \frac{\partial g(S_T)}{\partial S_0}$$

and note:

$$\frac{\partial V(S_0)}{\partial S_0} = e^{-r(T-t)} \int_0^\infty f(S_T) \frac{g'(S_T)}{g(S_T)} g(S_T) dS_T$$

Further:

$$g'(S_T) = -\psi(S_T) \frac{\partial \psi}{\partial S_0} \frac{e^{-.5\psi^2(S_T)}}{\sigma \sqrt{2\Pi(T-t)}}$$

Thus:

$$\frac{g'(S_T)}{g(S_T)} = -\psi(S_T)\frac{\partial\psi}{\partial S_0} = \frac{\ln(\frac{S_T}{S_0}) - (r - .5\sigma^2)(T - t)}{S_0\sigma^2(T - t)}$$

Further, note that

$$\frac{\ln(\frac{S_T}{S_0}) - (r - .5\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}}$$

is a standard normal variate (Z, say), so:

$$\frac{g'(S_T)}{g(S_T)} = \frac{Z}{S_0 \sigma \sqrt{(T-t)}}$$

Recalling that for a call (as an example)  $f(S_T) = \max[S_T(Z) - K, 0]$ , this all implies:

$$\frac{\partial V(S_0)}{\partial S_0} = e^{-r(T-t)} \int_0^\infty \max[S_T(Z) - K, 0] \frac{Z}{S_0 \sigma \sqrt{(T-t)}} \frac{e^{-.5Z^2}}{\sqrt{2\Pi}} dZ$$

This implies an estimate of Delta:

$$\tilde{\Delta} = \frac{1}{N} e^{-r(T-t)} \sum_{i=1}^{N} \max[S_T(Z_i) - K, 0] \frac{Z_i}{S_0 \sigma \sqrt{(T-t)}}$$

For path dependent options, things are only slightly more complicated when you recognize that:

$$g(S_1, S_2, \dots, S_T | S_0) = g(S_1 | S_0) g(S_2 | S_1) \dots g(S_T | S_{T-1})$$

where each of these conditional distributions is lognormal. Note that only  $g(S_1|S_0)$  depends on  $S_0$ , so:

$$\frac{\partial g(S_1, S_2, \dots, S_T | S_0)}{\partial S_0} \frac{1}{g(S_1, \dots, S_T | S_0)} = \frac{\partial g(S_1 | S_0)}{\partial S_0} \frac{1}{g(S_1 | S_0)} = \frac{Z(1)}{S_0 \sigma \sqrt{(t_1 - t_0)}}$$

where the last equality can be derived the same way as we did above, and here Z(1) refers to the random variable for the period  $t_1 - t_0$ . Thus:

$$\tilde{\Delta}_{Asian} = \frac{1}{N} e^{-r(T-t)} \sum_{i=1}^{N} \max[\bar{S}(\mathbf{Z}_i) - K, 0] \frac{Z_i(1)}{S_0 \sigma \sqrt{(T-t)}}$$

where  $\mathbf{Z}_i$  is the vector of shocks on simulation run *i* (i.e.,  $[Z_i(1), Z_i(2), \ldots, Z_i(T)]$ ).