FUTURES AND OPTIONS

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Basics of Forwards and Futures

A forward contract is an agreement between a buyer and a seller to transfer ownership of some asset or commodity ("the underlying") at an agreed upon price at an agreed upon date in the future.

A forward contract is a promise to engage in a transaction at some later date.

The forward contract specifies the characteristics of the underlying. For example, for a commodity, it specifies the type of commodity (e.g., silver), the quality of the commodity (e.g., 99.9 percent pure silver), the location of delivery, the time of delivery, and the quantity to be delivered.

The primary use of a forward contract is to lock in the price at which one buys or sells a particular good in the future. This implies that the contract can be utilized to manage price risk.

Most forward contracts are traded in the "over the counter" (OTC) market.

Some forward contracts are traded on organized exchanges such as the Chicago Board of Trade or the New York Mercantile Exchange. These exchange traded contracts are called "futures contracts."

Forward contracts traded OTC can be customized to suit the needs of the transacting parties. Exchange traded contracts are standardized. This enhances liquidity.

Performance on futures contracts are guaranteed by third parties (brokers and the clearinghouse.) Performance on OTC forwards is not guaranteed. The quality of the contractual promise is only as reliable as the firm making it.

Forwards and futures are traded on a wide variety of commodities and financial assets.

Commodity futures and forwards are traded on agricultural products (corn, soybeans, wheat, cattle, hogs, pork bellies); precious metals (silver, gold, platinum, palladium); industrial metals (copper, lead, zinc, aluminum, tin, nickel); forest products (lumber and pulp); and energy products (crude oil, gasoline, heating oil, natural gas, electricity).

Financial futures and forwards are traded on stock indices (S&P 500, Dow Jones Industrials, foreign indices); government bonds (US Treasury bonds, US Treasury notes, foreign government bonds); and interest rates (Eurodollars, EuroEuros).

More recently, forward/futures trading has begun on weather and credit risk. These are (no pun intended) the hottest areas in derivatives development.

The Uses of Derivative Markets

- Derivatives markets serve to shift risk.
- Hedgers use derivatives to reduce risk exposure. For instance, a refiner can lock in costs and revenues by buying crude oil futures and selling oil and gasoline futures.
- Speculators use derivatives to increase risk exposure in the anticipation of making a profit.
- Thus, derivatives markets facilitate the shifting of risk from those who bear it at a high cost (the risk averse) to those who bear it at a low cost (the risk tolerant).
- Speculators perform a valuable service by absorbing risk from hedgers. In return, they receive a reward—a risk premium. The risk premium is the <u>expected</u> profit on a derivatives transaction. Speculators may win or lose in any given trade, but on average speculators expect to profit.
- The risk premium is also the cost of hedging.

Commodity Derivatives: Precious Metals

Cash and Carry Arbitrage

Call S_t the spot price of a commodity (silver, for instance) at time t. Moreover, $F_{t,T}$ is the futures price at *t* for delivery at time *T*, r is the riskless interest rate between time *t* and *T*, and s is the cost of storage between *t* and *T*. Assume that there is no benefit of holding inventories of the commodity (i.e., the rental/lease rate is zero). Consider the following set of transactions. Buy the spot commodity at *t*, store it until *T*. Finance this purchase and storage with borrowing. Sell the futures contract for delivery at *T*. Consider the cash flows from these transactions at dates *t* and *T*.

TRANSACTION	Date *t*	Date *T*
Buy spot	$-S_t$	S_T
Borrow	$S_t + s$	$-e^{r(T-t)}[S_t + s]$
Sell Futures	0	$F_{t,T} - S_T$
Pay storage	-*s*	0
Net Cash Flows	0	$F_{t,T} - e^{r(T-t)}[S_t + s]$

All of the prices that determine net cash flows at T are known and fixed as of t. Therefore, this transaction is riskless. Since this transaction involves zero investment at t, in order to avoid the existence of an arbitrage opportunity it must be the case that the cash flows at T are identically 0. Therefore, in order to prevent arbitrage, the following relation between spot and futures prices, interest rates, and storage charges must hold at t:

$$F_{t,T} = e^{r(T-t)}[S_t + s]$$

Reverse Cash and Carry Arbitrage

Cash and carry arbitrage involves buying the spot, borrowing money, and selling futures. Reverse cash and carry arbitrage involves selling the spot short, investing the proceeds, and buying futures. In order to short sell the spot asset, an investor borrows the asset and sells it on the spot market. The asset borrower must purchase the asset at *T* in order to return it to the lender. The cash flows from this transaction are:

TRANSACTION	Date t	Date T
Sell spot	$S_t + s$	$-S_T$
Lend	$-(S_t + s)$	$e^{r(T-t)}[S_t + s]$
Buy Futures	0	$S_T - F_{t,T}$
Net Cash Flows	0	$e^{r(T-t)}[S_t + s] - F_{t,T}$

Note that this implies the same arbitrage relation as cash and carry arbitrage.

Note: This analysis assumes that the arbitrageur holds inventories of the commodity. If he doesn't, he does not save on storage charges by selling the commodity. This drives a wedge (equal to the cost of storage) between the cash-and-carry and reverse-cash-and-carry arbitrage restrictions.

Implied Repo Rates

Arbitrage restrictions define relations between spot and futures prices and interest rates that must hold in order to prevent traders from earning riskless profits with no investment. Since there are many futures markets, in order to identify arbitrage opportunities in several markets simultaneously, it is convenient to convert spot-futures price relations into a common variable. Since for a given trader the same interest rate should apply to any arbitrage transaction, regardless of whether the transaction is in gold, silver, Treasury bonds or corn, the obvious common variable is an interest rate. Therefore, traders use spot and futures price to calculate an implied interest rate. This is commonly called the implied repurchase ("repo") rate, because the repo rate represents the rate at which most large traders can borrow or lend.

To calculate an implied repo rate, take natural logarithms of the basic arbitrage expression. This implies:

$$\ln F_{t,T} = r(T-t) + \ln[S_t + s]$$

Simplifying this expression, and recognizing that the difference between two logs equals the log of their ratio, produces:

$$\ln[\frac{F_{t,T}}{S_t + s}] / (T-t) = r^i$$

This is the interest rate implied by spot and futures prices. If this implied repo rate is lower than the actual repo rate at which a trader can borrow/lend, the trader can borrow cheaply through the futures market, and lend through the repo market. Conversely, if the implied repo rate exceeds the rate at which the trader can borrow/lend through the money market, he should lend through the futures market and borrow through the repo market. Moreover, by comparing implied repo rates from futures on different commodities (e.g., gold vs. S&P 500) a trader can identify cheap borrowing and rich lending opportunities.

The Pricing of Energy and Industrial Metal Futures: Spread Relations

Precious metal futures are pretty straightforward. Usually they sell at near full carry (i.e., in contango). Lease rates are typically small, and there is an active market for leasing precious metal inventories.

Pricing energy and industrial metal futures is a little more complicated. In particular, the relations between spot and futures prices, and between nearby and deferred futures prices are much more complex than is the case with precious metals.

For example, sometimes oil, heating oil, or copper trade at nearly full carry. Other times, the futures prices are far below spot prices--that is, the market is in "backwardation." Moreover, the market can move from contango to backwardation to contango very quickly.

This raises the question: what determines the spreads between futures and spot prices for energy products and industrial metals?

The so-called "theory of storage" provides the answer to this question. In essence, this theory states that futures prices serve to ensure that consumption of these commodities is distributed efficiently over time.

Consider two situations:

Case 1. Supplies of the commodity (e.g., copper) are abundant. Delivery warehouses are full. Producers are operating with excess capacity.

Under these conditions, it makes sense to store some of the commodity. Abundant supplies are on hand, and to consume them all would glut the market today, and perhaps leave us exposed to a shortage in the future.

Traders will store the commodity (i.e., hold inventories) if and only if this is profitable. By storing (rather than selling) the commodity, the trader gives up the spot price (which he could capture by selling the commodity), and incurs interest and warehousing expenses. Storage is profitable only if futures/forward prices are above the spot price by the costs of interest and storage. That is, storage is profitable only if the market is at a carry/contango. If stores are huge, the market must be at full carry.

Case 2. Current supplies are very scarce relative to expected future production and supplies. That is, there is a temporary shortage. Producers are operating at or near capacity.

In these circumstances, it is foolish to store the commodity--it is scarce today relative to expected future supplies. Therefore, we should consume available supplies and hold close to zero inventories. (This exhaustion of inventories is sometimes called a "stock-out.")

Since the commodity is scarce today relative to what we expect in the future, the current spot price may be above the forward price. That is, the spot price must rise as high as necessary to ration the limited supplies. Moreover, there is no need to reward storage. Therefore, the market need not exhibit a carry. Thus, backwardation is possible.

There are some implications of this analysis:

Implication 1. The spread between spot and futures prices, adjusted for carrying costs, should depend upon the stocks held in inventory. When stocks are low, the market should be in backwardation; when stocks are high, the market should be at nearly full carry.

The industrial metals markets illustrate this clearly.

The theory also has implications for the volatility of prices, and the correlation between spot and forward/future prices. This is of great importance in risk management and option pricing applications.

Implication 2. When stocks are short prices should be more volatile. For those who are familiar with supply and demand analysis, a shortage implies that the supply curve is inelastic, i.e., steep. The steeper the supply curve, the more volatile spot prices. Since shortages are associated with backwardation, we expect very volatile prices in an inverted market (i.e., a market with backwardation), and less volatile prices when the market shows a carry.

Implication 3. Again calling upon supply and demand analysis, we know that short run supply curves are less elastic than long run supply curves. This implies that spot prices should be more volatile than forward/futures prices. Moreover, the difference in short run and long run supply elasticities should be greatest, the shorter are supplies today. Thus, the difference between spot volatility and forward volatility should rise as the market moves towards backwardation. Furthermore, this difference should be greater, the longer the time to delivery on the deferred contract.

Implication 4. When a stockout occurs, the cash-and-carry arbitrage link between spot and forward prices is broken. Under these conditions, spot and forward prices can move independently. When stocks are abundant, arbitrage ensures that spot and futures prices move together--the futures-spot spread equals the cost of carry. Thus, spot and forward/futures prices should be highly correlated when the market is at a carry, but may exhibit very low correlation when the market is in backwardation. Moreover, since a stockout is more likely, the longer the time to expiration of the deferred forward, the correlation between spot and forward/future prices should be lower, the greater the maturity of the forward/future.

Squeezes, Hugs, and Corners

Another factor which affects spot-futures spreads is a manipulation. A manipulation--sometimes referred to as a squeeze or corner, or a "hug" for a mild manipulation--occurs when a single trader accumulates a long futures/forward position that is larger than the physical supplies that can be delivered economically against these contracts. Rather than incur large costs to acquire deliverable supplies, as contract expiration approaches shorts are willing to buy back their contracts from the long at a premium.

This causes a unique pattern in prices. The price of the expiring future/forward that is being cornered rises--sometimes precipitously--relative to the deferred futures/forward price. Since the expiring future/forward and the spot price must converge during the delivery period, this means that the spot price rises relative to the deferred forward/future too. As soon as shorts close out their positions, the spot price collapses. The effect of final liquidation of a corner on spot prices is sometimes called the "burying the corpse effect." A sharp increase in shipments to delivery warehouses also occurs.

Squeezes/corners are not unknown in commodity derivative markets. Exchanges and governments attempt to prevent or deter them, but they still occur from time to time. The experience of the zinc market in 1989 provides a good illustration of the effects of a squeeze on spreads.

Moral of the story: When trading commodities, if you are short you must always be aware of the possibility of a manipulation. Monitor futures market and cash market activity continuously to make sure that you are not unexpectedly caught in a squeeze.

Non-Storable Commodities

- There are also futures and forward contracts traded on non-storable commodities.

- These include: electricity, weather, bandwidth, live animals (e.g., hogs and cattle).

- Non-storability has a big impact on price dynamics and forward pricing.

- Storage mitigates price volatility—inventories are accumulated when demand is low and supply is high, thereby reducing the magnitude of price declines under these conditions, and are drawn from when demand is high and supply is low, thereby mitigating the magnitude of price increases.

- Without storage, inventories cannot "smooth" the effects of supply and demand shocks.

- This implies that prices of non-storables—notably electricity—can be **extremely** volatile.

Forward Prices for Non-Storables

- Due to non-storability, cash-and-carry arbitrage is impossible—you can't hold an inventory of electricity from the nighttime to the afternoon.

- This contributes to considerable intra-day variation in prices for non-storables with systematic intra-day variation in demand or supply.

- It also makes cash-and-carry arbitrage pricing methods—"preference free" pricing techniques—impossible for non-storables.

- Therefore, estimating forward prices (and forward curves) for non-storables must take a different tack.

- Forward price=expected spot price + risk premium

Electricity Forward Pricing

- For electricity, for markets in which good spot price and demand (load) data are available, such as PJM, California, or Australia, can estimate expected spot prices using traditional statistical techniques.
- Statistical distribution of demand can be estimated accurately.
- Use observed spot-price/load data to estimate a function that relates prices to load.
- Combine demand distribution and spot price/load relation to estimate an expected spot price.
- Estimating risk premium is trickier—need to use market forward price data.
- Essential to take risk premium into account when pricing power forwards because this risk premium is huge, especially on-peak.

Power Derivatives Markets

- Power derivatives markets have grown, albeit somewhat slower than had been anticipated in the mid-1990s.

- Virtually all power forward and options trading done over-the-counter.

- Exchange markets have languished.

- Heterogeneity of electricity (especially locational differences) have impeded development of liquid power forward markets—heterogeneity fragments liquidity.

- Credit issues (note effects of 1998 Midwest price spike, California crisis) have also impeded market development.

- Finally, lack of integration of financial and physical markets has impeded development.

- Until these issues are resolved, power derivatives trading may prove treacherous.

Weather Derivatives

- Weather derivatives are a new frontier in derivatives trading.

- Weather derivatives trading almost exclusively OTC, although there are exchange-listed contracts on the CME.

- Typical weather derivative product is based on heating- or cooling degree days.

- One heating degree day occurs when the low temperature is one degree below 65^o for one day. A cooling degree day occurs when the high temperature is one degree above 65^o for one day.

- Most weather derivatives are heating or cooling degree day options. For instance, you could have a contract that pays $1 per cooling degree day times the difference between total cooling degree days in Chicago in July, 2002 and 250, if that difference is positive and zero if it is not.

- Weather derivatives can be used to manage quantity risk. For instance, the quantity of power or natural gas sold by an energy company depends on weather conditions. So does the price. Can use weather derivatives to manage this risk.

- Retailers can use them to manage risks due to weather. Retail sales for certain products are very sensitive to weather.

- Although degree-day options are the most common, weather derivatives can be based on rainfall, snowfall, or any other measurable weather variable.

- Like power derivatives, due to lack of storability arbitrage-based pricing of weather derivatives not possible. Need to utilize historical data on weather (corrected for global warming???) to estimate expected payoffs, then adjust by a risk premium.

Determining the Risk Premium

- For storables, our arbitrage analysis shows that the risk premium is irrelevant to determining the relation between spot and forward prices. That is, for these goods we can use "risk preference free" pricing to determine these relative prices. However, even for storables the risk premium affects the level of futures and spot prices, and their average movements through time.

- The earliest theory of the risk premium is due to Keynes. Keynes posited that hedgers are typically short futures. That is, they are typically holders of inventories of a commodity (e.g., corn) and they sell futures as a hedge.

- Since hedgers are net short, speculators must be net buyers in equilibrium (since total buys=total sells). Speculators will not absorb risk unless they are rewarded by profiting on average. Buying futures is profitable on average if futures prices rise on average. That is, speculators will enter the market only if the futures price is below the spot price expected at contract expiration. In this theory, the futures price tends to "drift up" over time—the speed of the drift measures the risk premium.

- In Keynes’ theory, futures prices are “downward biased.”

- In Keynes’ theory, this downward bias makes short hedging costly (since short hedgers lose on average) so they will tend to hedge less than 100 percent of their risk.

- If long hedgers outnumber short hedgers (at a futures price equal to the expected spot price) futures prices must be upward biased to attract speculative interest.

- Thus, net hedging interest determines whether futures prices are upward or downward biased.

- For some markets (exchange traded futures) there is data on net hedging interest in the form of the CFTC’s “Commitment of Traders Reports” available on the www.

The Magnitude of the Risk Premium

- The foregoing implies that the size of the risk premium depends on (a) net hedging demand at a price equal to the expected spot price, and (b) the risk aversion of speculators.

- The greater the hedging imbalance, the greater the risk premium (in absolute value).

- The more risk averse the speculators, the greater the risk premium (in absolute value).

- The risk aversion of speculators depends on (a) how well the futures market is integrated with the broader financial markets, and (b) the correlation between the futures price and movements in the market portfolio.

- Usually it is the case that risk premia will be smaller if the futures market is well integrated with broader financial markets because integration makes it possible for diversified speculators to participate in the market; diversification reduces speculator risk exposure.

- CAPM-type models imply that the higher the beta between the futures price and the overall market, the greater the upward drift in the futures price.

Estimating the Risk Premium

- The risk premium is the difference between the futures price and the expected spot price. Also, the risk premium affects the expected change in the futures price.

- Thus, estimation of the risk premium requires either estimation of the expected spot price, or estimation of the drift in futures prices.

- Since the risk premium affects both the costs of hedging and the benefits of speculation, both require estimation of expected spot prices. Thus, spot price forecasting is an important part of hedging and speculation.

- For most goods and commodities it is hard to estimate expected spot prices with accuracy. Electricity and weather may be exceptions.

Credit Derivatives

- Credit derivatives are another new type of financial contract. This is a rapidly growing market.

- Credit derivatives allow firms to hedge the risks of default on loans. For example, a bank that loans money to a customer loses if that customer defaults. The bank may be able to sell-off some portion of the credit risk without selling the loan, by entering into a credit derivatives transaction.

- Almost all credit derivatives are traded over the counter.

- There are a variety of credit derivative products in wide use.

Credit Event Derivatives

- A credit event derivative between party A and party B involves (a) a periodic payment from A to B, (b) the definition of a “credit event”, and (c) a payment from B to A if a “credit event” occurs.

- A credit event may be a bankruptcy or ratings downgrade, for instance. Although defining a credit event seems straightforward, it is not. The events in Russia in 1998 provide an illustration of the difficulties of defining a credit event.

- A may be a bank that wants to reduce its exposure to default by a particular borrower. B may be an insurance company or other financial firm that is willing to bear default risk.

- The size of the payment made in the event of a credit event is related to the impact of the event on the value of the loan.

- The size of periodic payment depends on the price the market charges to bear credit risk. Yield spreads between bonds of differing credit risk measure the market price of credit risk. Thus, the periodic payment should be related to yield spreads.

Total Return Swaps

- Total return swaps are another common type of credit derivative.

- In a total return swap, A and B exchange payments, where the magnitude of the payments swapped depend on the total returns on instruments of different credit-worthiness.

- For instance, the swap may involve A paying B the total return (interest plus capital gain/loss) on a BBB bond, and B paying A the LIBOR rate.

- This would make sense for A if he owned BBB bonds and didn’t want to bear the credit risk. In essence, this deal passes the credit risk to B without selling the actual bond.

Why Use Credit Derivatives?

- It seems somewhat weird that firms would alter credit risk exposures through credit derivatives—why don't they just trade the underlying loans? That is, why would a bank that wants to reduce credit exposure enter a total return swap instead of just selling off credit-risky loans?

- Taxes, accounting, and regulatory arbitrage. Sales of loans may have adverse impact on taxes or reported earnings or the balance sheet. For instance, a gain or loss must be recognized on sale for tax or accounting purposes, but may not be recognized if credit risk is transferred through a credit derivatives transaction. Also, some intermediaries may operate under regulations that limit their ability to purchase below-investment grade bonds, but that do not limit their ability to use credit derivatives.

- Liquidity. Credit derivatives may be more liquid than the underlying securities. This is typically the case since most credit derivatives have shorter maturities (e.g., one year) than the underlying securities (e.g., five years). Information asymmetries are plausibly smaller for shorter term instruments, making them more liquid

Eurodollar Futures and their European Cousins

Eurodollar futures contracts traded on the CME are the most heavily traded futures contracts in the world. The contract is cash settled--shorts don't deliver an ED deposit at expiration. Instead, on the second London business day before the third Wednesday of the contract month (March, June, September, or December) the CME calculates the average LIBOR rate. The mark-to-market price of the contract on this day is set equal to:

$$V = 10000(100 - .25R)$$

where R is the annualized LIBOR rate at settlement. The .25 multiplying the LIBOR rate converts the annual rate into a quarterly rate. Note that the contract price fluctuates inversely with interest rates (just like a bond price). A one bp increase in the interest rate leads to a $25 decrease in contract value.

Prior to expiration, the contract price is:

$$10000[100 - .25(100 - F)]$$

where F is the futures price on the day in question. This price is used to determine margin payments. Note that every .01 change in the futures price results in a $25.00 change in contract value. This is the price value of a basis point (PVBP) for the EDF.

There are currently related countracts in Europe on short term German, French, British, Spanish, and Dutch interest rates.

Eurodollar Futures Arbitrage

At contract expiration at *T*, the ED futures price is $100(1-4r^{T}_{T,T+90})$, where this interest rate is not annualized, but quarterly. Prior to *T*, the ED futures price is $100(1-4r^*)$ where r^* is the interest rate currently embedded in the futures price.

	Date t	Date T	Date $T+90$
Lend k dollars, with maturity $T+90$	$-k$	0	$k(1+r^{t}_{t,T+90})$
Borrow k, with maturity T.	k	$-k(1+r^{t}_{t,T})$	0
Short ED futures	0	$r^{T}_{T,T+90}-r^*$	0
Reinvest ED profits	0	$-(r^{T}_{T,T+90}-r^*)$	$(1+r^{T}_{T,T+90})(r^{T}_{T,T+90}-r^*)$
Borrow 1 at T with maturity $T+90$	0	1	$-(1+r^{T}_{T,T+90})$

We choose *k* such that profits at *T+90* are approximately equal to zero. We can't make profits riskless (i.e., we can't make them identically equal to 0 because we don't know the rate at which we will reinvest our futures profits/losses.

Formally, after doing a little algebra, profits at *T+90* are:

$$k(1+r^{t}_{t,T+90})+r^{T}_{T,T+90}-r*+r^{T}_{T,T+90}r^{T}_{T,T+90}-r^{T}_{T,T+90}r*-1-r^{T}_{T,T+90}$$

The two terms in the middle of this expression represent the difference between the product of two interest rates. This is small relative to the level of interest rates, so we will ignore it. Note, however, that this difference will seldom equal zero, and is unpredictable as of time *t*. Therefore, we cannot construct a perfect arbitrage because of the reinvestment risk on our ED futures. Thus, we get:

$$k(1+r^{t}_{t,T+90})=1+r*$$

Solving for k gives:

$$k=\frac{1+r*}{1+r^{t}_{t,T+90}}$$

Now turn to period *T*. Our net cash flows then are:

$$1 - \frac{1 + r^*}{1 + r^t_{t,T+90}}(1 + r^t_{t,T})$$

Note this cash flow is riskless as of *t*. In order for these cash flows to equal 0, the following expression must hold:

$$1 + r^t_{t,T+90} = (1 + r^t_{t,T})(1 + r^*)$$

Note that by the definition of a forward rate of interest, this expression implies that r^* is the forward rate of interest at *t* for the period *T* to *T+90*. That is, $r^* = r^t_{T,T+90}$

This is the basic no-arbitrage expression for ED futures because by substituting for *r** we get:

$$F_{t,T} = 100(1 - 4r^{t}_{T,T+90})$$

In essence, the arbitrage strategy involves long term lending financed by back-to-back short term borrowing. The ED futures position locks in the rate at which I can borrow over the period T to $T+90$. If I can borrow back-to-back over the period t to T+90 at a lower rate than I can lend over this period, there is an arbitrage opportunity. The no-arb expression essentially sets the borrowing and lending costs equal.

An example is useful here. Assume that date t is 10/15/01. Date T is 12/15/01. Date $T+90$ is 3/15/02. Assume that the annualized quarterly compounded interest rate over the period 10/15-12/15 is 4 percent, and the annualized quarterly compounded interest rate over the period 10/15/01-3/15/02 is 6 percent. Then

$$1+r_{t,T}^{t}=(1+\frac{61}{90}\frac{.04}{4})=1.0067$$

The term multiplying the interest rate in this expression reflects the fact that there are 61/90'ths of a quarter from t to T.

Similarly

$$1+r_{t,T+90}^{t}=(1+\frac{151}{90}\frac{.06}{4})=1.0252$$

Thus

$$1+r_{T,T+90}^{t}=\frac{1.0252}{1.0067}=1.01837$$

This implies $F_{t,T}=100[1-(4)(.01837)]=92.64$

If the actual ED futures price is higher than this value, one should short futures, lend from t to $T+90$, and borrow from t to T and T to $T+90$. If the actual ED futures price is lower than this value, execute the opposite transactions. Intuitively, if the EDF price is too high, one can lock in a low borrowing rate for the period T to $T+90$. Conversely, if the EDF price is too low, one can lock in a high lending rate from T to T+90.

Futures Rates, Forward Rates, and Convexity

The preceding analysis ignored one important institutional feature of futures contracts: Marking-to-market. The arbitrage table assumed that all cash flows on the futures contract occurred at expiration. This is true of forward contracts, but not futures contracts. Gains and losses on futures contracts are realized on a daily basis. If the futures price goes down (up) on a given day, the long (short) pays out this amount and the short (long) receives it. This process of daily settlement is a way of reducing default risk on futures contracts.

The aggregate amount of gain/loss incurred on a marked-to-market futures position equals the amount assumed in the arbitrage table, but the timing is different. This timing difference is important if there is a correlation between the futures price and interest rates. This timing difference can cause a difference between the no-arbitrage futures price, and the no-arbitrage forward price (which is what we actually derived in the ED futures analysis).

Case 1. Interest rate changes are negatively correlated with futures price changes. This is the case relevant for interest rate futures such as the ED future.

Let's compare the cash flows of a short position in a futures contract with those of a short position in a forward contract on the same underlying with the same expiration date. Let the futures price rise. Because interest rate changes are negatively correlated with futures price changes, this futures price increase is typically associated with a fall in interest rates. Thus, although the short must pay out losses (because the futures price rose) he can finance these at a low interest rate. Now consider a fall in futures prices. This is typically associated with a rise in interest rates under the proposed scenario. Thus, the short receives gains and can invest these at a higher interest rate.

The situation for the long is reversed--he can invest his mark-to-market gains at a low interest rate and must borrow at a high rate to finance his mark-to-market losses.

The holder of a forward contract receives no mark-to-market cash flows, so the correlation between price changes and interest rate changes doesn't matter to him regardless of whether he is long or short.

This implies that in this case there is an advantage (disadvantage) to being short (long) a futures contract when futures price changes and interest rate changes are negatively correlated. Competing to exploit this advantage, shorts bid down the futures price so that it is below the forward price. That is, shorts are willing to sell futures at a lower price than they are willing to sell forwards. **Thus, given the negative correlation, futures prices should be below forward prices**.

Case 2. Positive correlation between futures price changes and interest rate changes.

The analysis here is exactly the opposite of that in case 1. There is a benefit to *long* positions arising from marking-to-market. Thus, longs are willing to buy futures at a higher price than they are willing to buy forwards. **Thus, given a positive correlation between futures price changes and interest rate changes, futures prices should be above forward prices**.

This difference between futures and forward prices resulting from marking-to-market is essentially attributable to ***convexity***.

The size of the difference between forward and futures prices depends upon a) the size of the correlation between futures price changes and interest rate changes, and b) the volatilities of futures price changes and interest rate changes.

Holding volatilities constant, the bigger the correlation (in absolute value) the greater the difference because the party benefiting from the correlation can do so more often.

Holding correlation constant, the bigger the volatilities, the greater the difference because the party benefiting from the correlation can invest bigger gains at better interest rates. (Remember, convexity is more valuable, the greater is volatility.)

The effect may be small for some commodities. For example, the correlation between oil price changes and interest rate changes is about .01 or less--here there is no big difference between futures and forward prices. For ED futures, however, the effect can be large as the correlation is usually above .95.

How to Value the Effect of Convexity Bias

This correlation/convexity effect is important for ED futures pricing and hedging. It is also important for swap pricing. This is true because the ED futures (and other short term interest rate futures) are a very transparent, real time source of information about the term structure, but one cannot use these prices *directly* to determine a forward yield curve because of the convexity bias: ED futures prices tell you the **futures** interest rates, but for pricing FRAs or swaps you need to know the **forward** interest rates. *Futures interest rates implied by ED futures are higher than forward interest rates because the futures price is biased downward by convexity*. To derive a forward yield curve from EDF prices you must adjust for this convexity.

How can you do this? Follow these steps:

1. Assume that there are T days to futures expiration. Then it is possible to show theoretically that the futures price bias is equal to:

$$B = \sum_{t=0}^{T} s_{F}(t) s_{z(T-t)}(t) r_{F,z(T-t)} \equiv \mathrm{cov}(\Delta \ln F(t), \Delta \ln z(t,T))$$

where $s_{F}(t)$ is the volatility of the futures price at time t, $s_{z(T-t)}(t)$ is the volatility on day t of a zero coupon bond expiring at time T, and $r_{F,z(T-t)}$ is the correlation between percentage changes in the futures price and percentage changes in the price of the zero. (Note that “volatility” means the daily standard deviation of percentage price changes.) The “cov” term is the covariance between daily percentage futures price changes and daily percentage zero coupon price changes.

2. Note that all of the parameters affecting bias should changes as time to expiration nears. For example, ED futures prices should become more volatile as time to expiration nears (i.e., *T-t* becomes small) because forward rates tend to be less volatile than spot rates. Also, zero coupon bond price volatility changes as *T-t* changes because a) spot rates are more volatile than forward rates, and b) the duration of the zero falls as expiration nears. The correlation should also change as time passes.

Given this systematic change in parameters, to estimate the bias break down the interval between today and *T* into subperiods. For example, for a ED futures contract with six months to expiration, break the six month period into two three month periods. (You can break up periods more finely if you wish.) Noting that the term in the summation is equivalent to the covariance between % futures price changes and % zero price changes, use historical data to estimate the *daily* covariance between percentage price changes of EDFs with between 6 and 3 months to expiration and the percentage price changes of zero coupon bonds with between 6 and 3 months to expiration. Call this "cov1". Next estimate the *daily* covariance between percentage price changes of EDFs with between 3 and 0 months to expiration and the percentage price changes of zero coupon bonds with between 3 and 0 months to maturity. Call this "cov2".

Call t_1 the number of days between today and three months from today. Call t_2 the number of days between three months from today and contract expiration. Your estimate of total bias is:

$$B = t_1 \operatorname{cov}1 + t_2 \operatorname{cov}2$$

Note that as time passes t_1 gets smaller, and your estimate of bias falls as a result.

Interest Rate Swaps

Swaps are over-the-counter agreements between two companies to exchange cash flows in the future according to a pre-arranged formula.

Most interest rate swaps are so-called "plain vanilla" swaps. One party of the swap ("A") makes a fixed payment periodically to the other party company B. Company B makes payments periodically. These payments vary with interest rates. If interest rates rise, B pays more to A; if interest rates fall, B pays A less.

In order to economize on payments, cash flows are netted. That is, if A owes B 10, and B owes A 7, A simply pays B 3.

The "floating" side of most swaps is linked to the LIBOR rate. For example, party B (who pays floating) may be obligated to pay A the LIBOR rate x some notional principal value every six months.

For purposes of illustration, make the following assumptions:

1. The nominal principal value of the swap is Q. Note: this principal never changes hands in an interest rate swap; it is merely used to determine the interest payments made by A and B.

2. A agrees to pay B k every six months for 5 years.

3. Every six months, B agrees to pay A the 6 month LIBOR rate at t-.5 times Q. Thus, if the LIBOR rate (annualized) at time t-.5 is 10 percent, and Q is \$10 mm, then B pays A (.5)(.1)(\$10 mm)=\$50,000 at t. Call $R_{t-.5}$ the annualized 6 month LIBOR rate at t-.5.

Thus, every 6 months the payoff to B (who pays floating) is:

$$k - .5R_{t-.5}Q$$

Note that this is like the payoff to a long interest rate forward contract. If interest rates fall below $\frac{2k}{Q}$ at *t-.5* then B receives a cash inflow at *t*; if interest rates rise above this level, B pays cash out to A.

Since A and B swap cash flows in this fashion every six months for 5 years, the total swap contract is equivalent to a bundle of 10 forward contracts.

Given the yield curve at the time the swap is initiated (say, time 0), the fixed payment *k* is set in order to make the value of the bundle of forward contracts equal to 0. Formally, if $R^0_{t-.5,t}$ is the forward semi-annually compounded rate of interest, as of time 0, over the period *t-.5* to *t,* and r_t is the continually compounded interest rate used to discount cash flows received at *t* as of time 0, then the present value of the swap to B (the floating payer) is:

$$V = (k - .5R_0Q)e^{-.5r_1} + \sum_{i=2}^{i=10}(k - .5R^0_{.5(i-1),.5i}Q)e^{-.5ir_i}$$

When initiating the swap, the parties to the swap then choose *k* to set this expression to equal 0. (The value of the swap to A is -*V*.)

You can use the following logic to determine the fixed rate *k* at the initiation of a swap. Consider a swap with *N* payment dates. These payments occur *M* times per year--if pays occur semi-annually, for instance, *M*=2. First, use the forward yield curve to calculate the discount factors corresponding to each of the payment dates. Call D_j the discount factor corresponding to payment date j. Then the "fair" or "market" *k* is such that the present value of the cash flows on one DM (or dollar) of notional principal of the swap equals one DM. Formally:

$$1 = k(D_1 + D_2 + \ldots + D_N) + (1)(D_N)$$

(The last term reflects payment of the 1 DM of notional principal.)

Thus,

$$k = \frac{1 - D_N}{D_1 + D_2 + \ldots + D_N}$$

To convert this to an annual rate, multiply *k* by *M*. That is, $i = Mk$. This is the same as the par yield on a bond/note with the same maturity as the swap.

Swap rates are often quoted as a spread over some other interest rate. For example, USD interest rate swaps are usually quoted LIBOR vs. the US Treasury rate plus a spread. Thus, the fixed rate in a 5 year swap may equal the 5 year US T-note rate plus 75 bp. The "credit spread" is usually pretty stable within the day, so this quoting convention allows the quoted swap rate to respond continuously to interest rate movements.

Marking Swaps to Market

The valuation of outstanding swaps is straightforward. Break the swap up into 2 parts--the fixed rate part and the floating rate part.

You can value the fixed part like a bond. Consider a swap with *N* remaining fixed payments of *k*. The price of a zero coupon bond maturing at the date of payment *i* equals D_i. Then the value of the fixed side of the swap is:

$$V_{Fix} = \sum_{i=1}^{N} k D_i Q$$

The floating side is similarly easy to value. Note that on a payment date, the party receiving floating is indifferent between receiving the pre-determined payment on that date plus either a) receiving the floating payment over the remaining life of the swap, or b) receiving the notional principal today, and paying out the notional principal at the terminal date of the swap. Part b) is true because the party can invest the notional amount at the floating rate until maturity. Thus, on the payment date the value of the floating side must equal the value of alternative b):

$$V_{Float} = R + Q(1 - D_N)$$

where *R* is the floating payment due on that payment date. Thus, prior to the payment date, the floating side of the swap with *N* payments remaining is:

$$V_{Float} = D_1(R + Q) - D_N Q$$

The value of a short swap is simply $V_{Fix} - V_{Float}$.

LONG TERM INTEREST RATE FUTURES
The Forward Price of a Coupon Paying Bond

Call S_t the spot price (the full price including accrued interest) of a bond at time t. Moreover, $F_{t,T}$ is the futures price at *t* for delivery of this bond at time *T*, and *r* is the riskless interest rate between time *t* and. The bond pays a coupon equal to D at t_D. Also assume that interest rates are constant over time. (This ensures you can reinvest coupons at a known rate.) Consider a "cash and carry arbitrage" strategy:

TRANSACTION	Date *t*	Date t_D	Date *T*
Buy spot	$-S_t$	0	S_T
Borrow	S_t	0	$-e^{r(T-t)}S_t$
Sell Futures	0	0	$F_{t,T} - S_T$
Dividend Payment	0	D	0
Invest Dividend	0	-D	$e^{r(T-t_D)}D$

This analysis implies that net cash flows at the initial date are equal to 0, and at date T they equal:

$$F_{t,T} - e^{r(T-t)}S_t + e^{r(T-t_D)}D$$

Arbitrage then implies

$$F_{t,T} = e^{r(T-t)}S_t - e^{r(T-t_D)}D$$

A similar expression is relevant for a futures/forward on any asset that pays a cash flow. For example, for a dividend paying stock *D* can be interpreted as the dividend and *S* as the current price of the stock.

This analysis implies that the futures price equals the future value of the underlying price net of the future value of the cash flows the asset pays prior to contract expiration. This deduction of interim cash flows is necessary because the underlying price embeds the value of these cash flows, but the buyer of the forward contract does not receive them.

Note that the futures price exceeds the spot price by less than the interest rate factor.

One can calculate an “implied repo rate” from this expression. The implied repo rate is the short term interest rate that must prevail in the market to make arbitrage unprofitable.

$$r^i = \ln[\frac{F_{t,T} + e^{r(T-t_D)}D}{S_t}] / (T-t)$$

(Note that this expression is somewhat inconsistent because one has to put include an interest rate within the log expression. In order to solve for a single implied interest rate, one could use numerical techniques, but this is beyond the scope of this course.)

If the actual short term interest rate differs from the implied repo rate, an arbitrage opportunity exists. If the implied rate is below the actual rate, borrow through the futures and bond market by selling bonds spot, buying futures, and investing the proceeds at the repo rate. If the implied rate is above the actual rate, borrow at the repo rate and lend through the futures market by buying bonds spot and selling futures.

There are also some complications when the term structure of interest rates (i.e., the relation between interest rates and maturity) is not "flat." This expression must be modified if the term structure slopes up or down because the rate at which a trader can re-invest the coupons is different from the initial rate.

T-Bond Futures: Conversion Factors and Delivery Options

The Treasury Bond futures contracts traded on the CBoT allow shorts to deliver any bond with greater than 15 years to maturity (or call) against the contract. Since bonds can have very different prices due to differences in coupons or maturity, the exchange has devised a system to make it more economical to deliver a wide variety of bonds. In essence, the short receives higher proceeds if he delivers a high priced (high coupon, long maturity) bond than if he delivers a low priced one.

The exact system works as follows. Upon delivery of a given bond *i*, the short receives the following amount of money, called the invoice price:

$$P_i = CF_i F + AI_i$$

Here *CF* is the "conversion factor" of bond *i*, *F* is the futures settlement price on the day the short delivers, and *AI* gives the accrued interest on the bond.

The conversion factor is equal to the value of one dollar of face amount of the bond as of the first day of the delivery month under the assumption that the bond yields 6 percent (semi-annually compounded) to maturity.

Shorts have the option to choose which bond they will deliver. They will choose to deliver the bond for which the net proceeds (invoice price net of the full price of the bond) are largest. That is, they choose the bond *i* for which the following expression is largest:

$$P_i = CF_i F - B_i$$

where B_i is the *flat* price of the bond.

The bond that has the largest delivery value is called the "cheapest-to-deliver" ("CTD") bond. Note that if the foregoing expression were strictly positive for any bond, then an arbitrage opportunity would exist. Therefore, at the expiration of the futures contract the futures price equals:

$$F = \frac{B_i(t,T)}{CF_i}$$

where *i* is the CTD bond and $B_i(t,T)$ is the forward price of the bond at *t* for delivery at *T* (futures expiration).

Prior to expiration, it is possible to figure out the bond for which B_i / CF_i is smallest. This is the bond that is *currently* CTD. However, there is some possibility that some other bond may be CTD at expiration. This can occur because the relative prices of bonds change randomly over time. That is, prior to expiration, the futures contract is actually an option because the short can choose which bond to deliver. This option is an exotic--a call on the minimum of the *N* deliverable bonds, with a strike price of zero. It is possible to show that due to this option, the bond futures price is:

$$F_{t,T} = \sum_{i=1}^{N} w_i \frac{B_i(t,T)}{CF_i}$$

where the $w_i < 1$ are weights that add up to a number less than one. These weights are functions of all the bond prices and conversion factors, the volatilities of all the bond returns, and the correlations between bond returns. Note that the futures price must be less than the smallest B_i / CF_i because of the delivery option.

This option affects the nature of the futures contract as a hedging vehicle. If you want to match PVBPs, you need to know the PVBP of the future. This, in turn, depends upon the PVBPs of all the deliverable bonds. Specifically:

$$PVBP_F = \sum_{i=1}^{N} w_i \frac{PVBP_i(t)}{CF_i}$$

Note that the PVBP of the future can change over time because a) the PVBPs of individual bonds change, and b) the weights in the expression change. This can cause some funny things to happen. On one day, a high duration bond may receive a large weight and a low duration bond a low weight; a couple of days later, the low duration bond may have a high weight and the high duration bond a low weight. Thus, you need to monitor the PVBP of the future fairly closely to make sure that your hedge ratios remain correct.

There are other options embedded in the T-bond contract. The most important, and unusual, is the so-called “Wildcard” option. This option exists because the short can deliver up to 8pm at a settlement price set at 2pm on that day. Thus, if bond prices fall between 2pm and 8pm, it can be profitable to wait to deliver until 8pm. This option is very complex to value, especially due to its interaction with the “quality” option (i.e., the option to deliver the CTD bond).

Foreign Currency Cash and Carry Arbitrage

In determining the relation between foreign exchange spot and futures prices, it is necessary to expand the arbitrage table somewhat because we have to keep track of borrowing and lending in two currencies. Moreover, we have to take into account that we can borrow or lend in the foreign currency at rate r_f. Call S_t the spot price of Deutschmarks ("DM") in dollars. Thus, a spot price of .6471 means that each DM buys .6471 dollars.

	Time t		Time T	
	USD	DM	USD	DM
Buy $e^{-r_f(T-t)}$ DM	$-e^{-r_f(T-t)}S_t$	$e^{-r_f(T-t)}$	0	0
Lend DM	0	$-e^{-r_f(T-t)}$	0	1
Borrow USD	$e^{-r_f(T-t)}S_t$	0	$-e^{(r-r_f)(T-t)}S_t$	0
Sell DM Futures	0	0	$F_{t,T}$	-1
Net Cash Flows	0	0	$F_{t,T} - e^{(r-r_f)(T-t)}S_t$	0

To prevent arbitrage, the following expression must therefore hold:

$$F_{t,T} = e^{(r-r_f)(T-t)} S_t$$

Note that the fact that the foreign currency can be borrowed/lent at interest depresses the futures price relative to the spot price. Moreover, if the foreign interest rate is larger than the domestic interest rate, then the futures price is below the spot price.

Also recognize that you can use futures markets to borrow and lend foreign currencies. To borrow DM, undertake the following transactions: buy DM spot, borrow USD, sell DM futures.

You can also use intermarket spreads to borrow dollars through one FOREX futures market, and lend dollars through another. For example, if you borrow DM and sell them for dollars, and buy DM futures, this is equivalent to borrowing dollars. You can then lend dollars through, say, the JY futures market by using the USD you earned from the short sale of the DM by buying JY spot, lending JY, and selling JY futures. This is equivalent to borrowing DM, selling them for JY, lending the JY, buying DM futures, and selling JY futures. This can be a profitable transaction if the implied USD repo rates differ between the JY and DM markets.

OPTIONS BASICS

1. A call option gives the owner the right, but not the obligation, to buy the underlying asset (e.g., a stock, a bond, a currency, a futures contracts) at a fixed price. This fixed price is called the "strike price." Call options have a fixed expiration date. Time to expiration can range between days and years.

2. A put option gives the owner the right, but not the obligation to sell the underlying asset at a fixed price.

3. There are two basic types of options: European and American. The holder of a European option can exercise it only on the expiration date. That is, there is no early exercise for European options. In contrast, the holder of an American option can exercise it any time prior to the expiration date as well as on the expiration date itself. Most exchange traded options are American. Many over the counter options are European.

4. Options are traded on stocks at the Chicago Board Options Exchange ("CBOE"), the NYSE, the American Stock Exchange, the Pacific Stock Exchange and the Philadelphia Stock Exchange. CBOE is the oldest and largest of the options exchanges.

5. Options on futures are traded on most futures exchanges.

6. Options on currencies are traded on the Philadelphia Exchange.

7. OTC options markets are also important.

8. Many financial instruments have options embedded in them. For example, a mortgage gives the mortgagee the option to prepay at any time. A convertible bond embeds a call option.

ARBITRAGE RESTRICTIONS

Options prices must obey certain arbitrage restrictions.

1. Put-call parity. Consider *European* options on a non-dividend paying stock with price S_t. A position consisting of a call struck at *K* and a short put with strike *K* provides the same payoffs as a forward contract with a forward price equal to *K*. Both options expire at time *T*. We know that the fair (i.e., no arbitrage) forward price equals $e^{r(T-t)}S_t$. Thus, the present value of a forward contract with a forward price of *K* equals $S_t - e^{-r(T-t)}K$. Since the long call-short put position gives the same payoff as a forward contract, it must be the case that the value of this position equals the value of the forward contract. That is:

$$c(S_t, K, t, T) - p(S_t, K, t, T) = S_t - e^{-r(T-t)}K$$

This is called the put-call parity relation.

2. Given put and call prices, the stock price, the strike price, and the time to expiration, it is possible to solve the put-call parity expression for an implied interest rate in order to determine whether these prices present an arbitrage opportunity.

3. If the stock pays dividends prior to the expiration of the options, it is straightforward to modify the put-call parity expression to reflect this fact. Recall that in deriving put-call parity, we simply equated the value of the option position to the value of a forward contract with forward price *K*. Therefore, all that is necessary to adjust the formula for dividends is to use our no-arbitrage formula for a forward price on an asset paying a dividend. Here, the present value of such a forward contract is

$$S_t - e^{-r(t_D - t)}D - e^{-r(T-t)}K$$

where t_D is when the dividend is paid. Therefore,

$$c(S_t, K, t, T) - p(S_t, K, t, T) = S_t - e^{-r(t_D - t)}D - e^{-r(T-t)}K$$

EARLY EXERCISE OF OPTIONS

Early exercise has one clear disadvantage: by exercising an option, a trader gives up the value of the option over its remaining life. This value must be positive. Therefore, early exercise is desirable if and only if there is some off-setting benefit. There is only one possible source of such a benefit. Namely, there is a potential value to early exercise if this allows the owner of the option to receive cash flows earlier.

Case 1. Call on a non-dividend paying stock.

Assume the owner of a call with strike price K and time to expiration T exercises the option at $t<T$, and borrows money in order to pay the strike price. Then, at T, the wealth of the trader is $S_T - e^{r(T-t)}K$. If the trader had not exercised the option, his wealth at T would equal $\max[0, S_T - K]$. It is clear that the trader's wealth is greater if he does not exercise early. This is true for two reasons, first, if the price of the stock falls below the strike price between t and T, the trader won't exercise the option at T, and thus isn't "stuck" with a less valuable stock. Second, by deferring exercise, the call owner saves the interest on the strike price over the period t to T. Conclusion: Never exercise a call on a non-dividend paying stock early.

Case 2. Call on a dividend paying stock.

Assume the firm pays a dividend equal to *D* at *t*<*T*. If the call owner exercises at *t*, his wealth at *T* equals:

$$S_T - e^{r(T-t)}(K - D)$$

This may (but may not) exceed $\max[0, S_T - K]$ because by exercising the call early, the owner receives the dividend. Therefore, unlike the case with a call on a non-dividend paying stock, we can't say for certain that the payoffs to early exercise are always lower than the payoffs from exercise at expiration. If this dividend is large enough, it may exceed the interest on the strike price and the option value foregone from early exercise. Thus, you may exercise a call on a dividend paying stock early. If you do exercise early, you will only do so immediately before the payment of a dividend (i.e., on the cum dividend date).

If the future value of the dividend is smaller than the value of the interest paid on the strike price from t to T, early exercise will not be optimal. The value of this interest equals $e^{r(T-t)}K - K$. If $e^{r(T-t)}K - K > e^{r(T-t)}D$ then

$$S_T - e^{r(T-t)}(K - D) < S_T - K < \max[0, S_T - K]$$

and therefore early exercise is not profitable.

Intuitively, this means that if the dividend received by exercising early is not large enough to compensate the option holder for the interest incurred on the strike price due to early exercise, early exercise cannot be optimal.

Case 3. Put on a non-dividend paying stock.

Consider a trader who owns a put and a share of the underlying stock. If he exercises the put at $t<T$ and invests the strike proceeds, his wealth at t equals $e^{r(T-t)}K$. There are some values of the stock price such that this exceeds the trader's wealth at T if he does not exercise the put at t, $\max[0, K - S_T] + S_T$. By exercising early, the put owner receives cash flows (i.e., the proceeds from exercise) earlier; the interest earned on the strike price over this interval of time. This may compensate the trader for the gains he would earn by holding onto the stock if the price of the stock were to rise above the strike price between t and T. Early exercise of a put can be optimal at any time prior to expiration.

Case 4. Put on a dividend paying stock.

All else equal, dividend payments cause the price of the stock to decline. Therefore, if the put is not protected against dividend payments, dividends tend to induce the holder of a put to defer exercise in order to take advantage of this predictable price decline. It still may be the case, however, that the advantages of receiving the proceeds from exercise early (i.e., the interest earned on the strike price) offsets this effect.

Case 5. Puts and calls on futures contracts.

When the holder of an option on a futures contract exercises, she receives a cash payment equal to the difference between the strike price and the futures price at the time of exercise and a futures position. For example, if the futures price at exercise equals F_t, the holder of a put receives a cash payment equal to $K - F_t$ and a short futures position at the market price. The value of the short position equals 0. Early exercise of either a put or a call therefore leads to an acceleration of cash flows. Therefore, one cannot rule out early exercise of futures options.

Case 6. Calls on foreign currency.

Foreign currencies can be invested at interest; recall that this is equivalent to a continuous dividend yield. Therefore, there may be advantages to exercising a call on a foreign currency early in order to receive this cash flow over a longer period of time.

Early exercise requires a modification in the put-call parity expression. In particular, we can no longer derive an equality restriction, but only an inequality restriction.

First consider the case of American puts and calls on non-dividend paying stocks. We know that the value of an American call on a non-dividend paying stock equals the value of a European call on the same stock (with the same strike and time to expiration). Moreover, we know that an American put may be exercised prior to expiration. This option to exercise early has value, so an American put must be more valuable than a European put.

Therefore,

$$C(S_t, K, t, T) - P(S_t, K, t, T) < S_t - e^{-r(T-t)}K$$

Also remember that dividends reduce the value of an American call, and increase the value of an American put. Thus, this expression must hold for options on dividend paying stocks too.

A BINOMIAL MODEL OF STOCK PRICE MOVEMENTS

In order to derive a formula for stock option prices, we must first specify a model that describes how stock prices move.

Our first model is a so-called binomial model because at any point in time, it is assumed that the percentage change in the stock price--the stock "return"--can take only two values, u>1 (an "up" move) or d<1 (a "down" move). The probability of an up move equals q, and the probability of a down move equals 1-q. We will see that these actual probabilities are irrelevant to the pricing of options.

The binomial model divides time between the present and the expiration date of an option into discrete intervals of equal length. Each interval is Δt in length. By allowing more intervals between now and expiration, this interval becomes shorter. In the limit, with an infinite number of intervals, the length of an interval becomes vanishingly small, and equal to dt.

That is, if the stock price equals S today, it may equal either $uS > S$ or $dS < S$ at the end of the next interval of time.

It is possible to show that as the length of a time interval becomes vanishingly small, the distribution of stock prices is lognormal. We will utilize this fact in deriving the Black-Scholes formula for pricing options.

USING THE BINOMIAL MODEL TO PRICE AN OPTION

Consider a portfolio of a short position in a single call on a stock, and Δ shares of the stock underlying the call. Assume the value of the call in Δt units of time equals c_u if the stock price rises over this interval, and equals c_d if the stock price falls. Thus, at $t+\Delta t$ the value of the portfolio equals $\Delta uS - c_u$ if the stock price rises, and equals $\Delta dS - c_d$ if the stock price falls.

Note that we can choose Δ to make the value of the portfolio the same regardless of whether the stock price rises or falls.

Formally, choose Δ such that

$$\Delta uS - c_u = \Delta dS - c_d$$

or

$$\Delta = \frac{c_u - c_d}{S(u-d)}$$

Since the value of the portfolio doesn't depend upon the stock price change, the portfolio is riskless. Therefore, the return on the portfolio must equal the risk free rate. That is, if the value of the call today equals c, then:

$$e^{r\Delta t}(\Delta S - c) = u\Delta S - c_u = d\Delta S - c_d$$

Solving for c implies:

$$c = e^{-r\Delta t}[pc_u + (1-p)c_d]$$

where

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

Note that the value of the call at t is equal to the expected present value of the call at $t+\Delta t$, using p to measure the probability of an up move and (1-p) to measure the probability of the down move.

It is essential to recognize that p is not equal to q, the *true* probability of a stock price increase. Instead, p is the probability of an up move such that the stock's expected rate of return equals the risk free rate of return. This would occur in a market where traders are risk neutral.

This analysis implies that we can price options as if we are in a risk neutral world! Put differently, we don't need to know the expected return on a stock to price options on that stock. This is true because we can form a portfolio consisting of the stock and the option to eliminate all risk.

In order to determine the price of the call, we work backwards from the end of the binomial tree because at the end of the tree we know the values of the option.

This is best illustrated by an example. The following pages present a two stage binomial tree which assumes that the initial value of the stock price is 20, u=1.1, d=.9, and r=.12 (annualized, continuously compounded rate). Each time interval is a quarter of a year. The first page outlines what the values of the stock can take after 2 periods.

Given these values, we know that p=.6523.

Consider a call expiring in two periods with a strike price of 21. The value of the call in two periods after two up moves equals 24.2-21=3.2. The value of the call after one up move and one down move equals 0 because 19.8<21. Similarly, the value of the call after two down moves equals 0.

Consider the value of the call after one period if a single up move has taken place. Here $c_u = 3.2$ and $c_d = 0$. Thus, after one up move, the value of the call equals:

$$c = e^{-.12(.25)}[.6523 \times 3.2 + .3477 \times 0] = 2.0257$$

It is straightfoward to recognize that the value of the call after one period equals zero if the stock has moved down during that interval.

Now move back to the initial time, when the value of the stock equals 20. Here $c_u = 2.0257$ and $c_d = 0$. Thus, the value of the call at this time equals:

$$c = e^{-.12(.25)}[.6523 \times 2.0257 + .3477 \times 0] = 1.2823$$

In sum, we use backward induction repeatedly to value an option on an underlying stock. We go backwards because we know the payoffs to the option at the expiration date, and can therefore apply our formula repeatedly by proceeding from the end of time to the beginning.

The main choice you must make in establishing a binomial tree is for u and d. We choose these parameters such that the theoretical value of the variance of the stock given by the binomial model equals the actual value of the variance of the stock we are interested in.

Remember that the variance of a stock's return equals the expected value of the squared deviation between the realized return and the expected return. The expected return in our risk neutral model equals r. The return in an upmove equals u and the return in a down move equals d. Therefore, the variance equals:

$$p(u - e^{r\Delta t})^2 + (1 - p)(d - e^{r\Delta t})^2 =$$
$$p(1 - p)(u - d)^2 = s^2 \Delta t$$

where σ is the actual standard deviation of the stock's return.

The first equality follows from the fact that

$$Se^{r\Delta t} = puS + (1 - p)dS$$

We have already found the relation between p and u and d. Thus, we have one equation in two unknowns. We eliminate one unknown by choosing $d=1/u$.

If we solve all of this for u, we get: $u = e^{s\sqrt{\Delta t}}$ if Δt is small.

DERIVATION OF THE FORMULA FOR *p*.

First, define $a = e^{r\Delta t}$.

Then:

$$a\Delta S - u\Delta S + c_u = ac$$

Substituting for Δ implies:

$$\frac{a(c_u - c_d)}{u-d} - \frac{u(c_u - c_d)}{u-d} + c_u = ac$$

Gathering terms with c_u and c_d:

$$\frac{c_u}{u-d}(a - u + u - d) + \frac{c_d}{u-d}(u-a) = ac$$

$$\frac{a-d}{u-d}c_u + \frac{u-a}{u-d}c_d = ac$$

Define $p = (a-d)/(u-d)$ then note that

$$\frac{u-a}{u-d} = 1 - \frac{a-d}{u-d} = 1 - p$$

Thus,

$$ac = pc_u + (1 - p)c_d$$

Dividing both sides by *a* produces the expression presented earlier. It is important to note that if the probability of an upmove in the stock price equals *p*, then the expected return on the stock equals the risk free rate of return.

To see why, note that if the probability of an upmove equals *p*, the expected value of the stock next period equals:

$$\begin{aligned}
&S[pu + (1 - p)d] = \\
&S[p(u - d) + d] = \\
&S[\frac{a - d}{u - d}(u - d) + d] = \\
&aS
\end{aligned}$$

This implies that the stock earns an expected return equal to the risk free rate of interest.

Use of the Binomial Model

A major advantage of the binomial model is that it can be used to value American options for which early exercise is possible, and hence valuable. An example illustrates this.

First consider the example contained on the following page: an American put on a stock. (This is example 14.1 from Hull.)

In this example, σ=.40, T-t=.4167, r=.1, S=50, K=50. If we divide the time until expiration into 5 segments, we get Δt=.4167/5=.0833.

Given these values, we can determine p=.5076, and 1-p=.4924. Also, a=1.0084, and $e^{-r\Delta t} = 1/1.0084 = .9917$.

The important thing to do when "folding back" the binomial tree is to determine at each node whether early exercise is optimal. First consider node E. The stock price at this node equals 50, so the proceeds from early exercise equal 0. The value of the option if we don't exercise equals

[(.5076)(0)+(.4924)(5.45)]/1.0084=2.66.

If we do exercise at node E, we get less than if we wait. So we don't exercise early.

Things are different at node A, when the stock price equals 39.69. Here the proceeds from early exercise are 50-39.69=10.31. If we don't exercise, the option is worth:

$$[(.5076)(5.45)+(.4924)(14.64)]/1.0084=9.89.$$

At this node, we get more if we exercise early than if we don't. Early exercise is therefore optimal in this case. The important thing to remember is that we now use 10.31 as the value of the option at node A when we are calculating the option price at earlier points in the tree.

This demonstrates how we can use the binomial option to calculate American option prices.

Analytically, the binomial model is straightforward. In order to increase the accuracy of this approach, however, it is necessary to make Δt fairly small by increasing the number of time intervals. This can increase the cost of computing options prices using the binomial method.

Thus, when pricing an option, we face a trade off: the binomial method can handle early exercise easily, but is computationally cumbersome.

This raises the question, is there a more computationally tractable model?

The answer is yes: If we are willing to consider only European options, it is possible to produce an option pricing model that is very easy to use--the Black-Scholes model.

THE BLACK-SCHOLES MODEL FOR A NON-DIVIDEND PAYING STOCK

The Black-Scholes model is essentially the same as the binomial pricing model when the number of time intervals approaches infinity, i.e., as Δt becomes arbitrarily close to zero. This is sometimes called a "continuous time" model in contrast to the "discrete time" binomial model because we no longer divide the time line into several discrete periods, but instead consider time as a continuum.

It is possible to show that as the number of time steps approaches infinity, the return on the underlying stock obeys a normal distribution. The normal distribution is simply the well known bell shaped curve.

Recall that the return on a stock equals the percentage change in price on the stock. Also note that over a very small time interval dt the return on a stock equals:

$$\ln S_{t+dt} - \ln S_t$$

Thus, if the return on the stock in the continuous time world is normally distributed, then the stock price in the future is lognormally distributed.

It is also essential to remember that in valuing options we can assume that the expected return on the stock equals the risk free rate of interest. Again, this is because we can construct a portfolio including the stock and the option that is riskless.

This can be represented formally. If the stock price at T (which may be the expiration date of an option) is lognormally distributed, then we can write:

$$S_T = S_t e^{(r-.5s^2)(T-t)+s\sqrt{T-t}Z}$$

where Z is a normally distributed variable with expected value (i.e., mean) equal to 0.

(You can check that this expression is correct by taking the natural logs of both sides. You will find that the difference in the logs is normally distributed because Z is normally distributed.)

We can now value a European option that expires at T. Recall that the value of the option at t is the expected present value of the payoffs of the option at T, where we use the riskless interest rate to discount these payoffs. Also remember that any expected value is the sum of the possible payoffs multiplied by the probability of receiving a given payoff. In our analysis, the size of the payoff depends upon the realization of Z because Z determines the stock price at T. Moreover, because Z is normally distributed, the probability of observing any given Z is

$$n(Z) = \frac{e^{-.5Z^2}}{\sqrt{2\Pi}}$$

Note that the Π in this expression is the mathematical constant Pi=3.14159 . . .

We can use this information to value an option. Consider a call option with strike price K. We know that the option's payoff is positive if and only if:

$$S_T = S_t e^{(r-.5s^2)(T-t)+s\sqrt{T-t}Z} \geq K$$

There is a value of Z, call it Z*, such that this expression holds with equality. This is the "critical value" of Z: For larger Z, the call is in the money at expiration, for smaller Z, the option is out of the money. Therefore:

$$\ln S_t + (r-.5s^2)(T-t) + s\sqrt{T-t}Z^* = \ln K$$

Solving for Z* implies:

$$Z^* = \frac{\ln(K/S_t) - (r-.5s^2)(T-t)}{s\sqrt{T-t}}$$

Given this result, the expected present value of this option's payoffs (and hence its price) is:

$$c = e^{-r(T-t)}\{\int_{Z^*}^{\infty}[S_T(Z) - K]n(Z)dZ + \int_{-\infty}^{Z^*}0n(Z)dZ\}$$

This expression holds because the payoff to the option is 0 when Z<Z*.

Doing a little substitution, we get

$$c = e^{-r(T-t)} \int_{Z*}^{\infty} [S_t e^{(r-.5s^2)(T-t)+sZ\sqrt{T-t}} - K] \frac{e^{-.5Z^2}}{\sqrt{2\Pi}} dZ$$

Consider the exponentials. We can add the exponential terms to get the following exponent:

$$(r-.5s^2)(T-t) + sZ\sqrt{T-t} - .5Z^2 =$$

$$r(T-t) - .5[s^2(T-t) + Z^2 - 2sZ\sqrt{T-t}]$$

Define a new variable

$$y = Z - s\sqrt{T-t}$$

Note that y is normally distributed (because Z is). Moreover,

$$y^2 = s^2(T-t) + Z^2 - 2sZ\sqrt{T-t}$$

In addition, the call option is in the money if

$$y = Z - s\sqrt{T-t} \geq Z^* - s\sqrt{T-t}$$

$$y \geq Z^* - s\sqrt{T-t} =$$

$$\frac{\ln(K/S_t) - (r - .5s^2)(T-t)}{s\sqrt{T-t}} - s\sqrt{T-t} =$$

$$\frac{\ln(K/S_t) - (r + .5s^2)(T-t)}{s\sqrt{T-t}} \equiv y^*$$

Then we can rewrite the first term in our integral as

$$e^{-r(T-t)} \int_{y^*}^{\infty} S_t e^{r(T-t)} \frac{e^{-.5y^2}}{\sqrt{2\Pi}} dy =$$

$$\int_{-\infty}^{-y^*} S_t \frac{e^{-.5y^2}}{\sqrt{2\Pi}} dy$$

This is just equal to N(-y*), where N(x) gives the area under a normal distribution curve to the left of x. Thus, the first term in our integral is:

$$S_t N[\frac{\ln(S_t / K) + (r + .5s^2)(T-t)}{s\sqrt{T-t}}]$$

Now consider the second term in our integral:

$$e^{-r(T-t)} K \int_{Z*}^{\infty} n(Z) dZ =$$

$$e^{-r(T-t)} K \int_{-\infty}^{-Z*} n(Z) dZ =$$

$$e^{-r(T-t)} KN(-Z*) =$$

$$e^{-r(T-t)} KN[\frac{\ln(S_t / K) + (r - .5s^{2})(T-t)}{s\sqrt{T-t}}]$$

Putting the two terms together produces the Black-Scholes formula for the call:

$$c = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

where

$$d_1 = \frac{\ln(S_t / K) + (r + .5 s^2)(T-t)}{s\sqrt{T-t}}$$

and

$$d_2 = d_1 - s\sqrt{T-t}$$

We can then use put-call parity to get the value of a put. Recall:

$$c - S_t + e^{-r(T-t)} K = p$$

Then

$$p = S_t(N(d_1) - 1) + e^{-r(T-t)} K(1 - N(d_2))$$

Note that $N(x) = 1 - N(-x)$. Therefore,

$$p = e^{-r(T-t)} K N(-d_2) - S_t N(-d_1)$$

OPTIONS ON OTHER ASSETS

One can rewrite the Black-Scholes formula for a call as follows:

$$c = e^{-r(T-t)}[e^{r(T-t)}S_t N(d_1) - KN(d_2)]$$

Note that the term multiplying $N(d_1)$ is the forward price of a non-dividend paying stock.

One can also rewrite the term inside the normal function in the model as:

$$d_1 = \frac{\ln(Se^{r(T-t)} / K) + .5s^2(T-t)}{s\sqrt{T-t}}$$

Note that the numerator in the logarithm term is also the forward price of a non-dividend paying stock. In general, it is possible to show that to find the value of a call on some other type of asset, one simply uses the forward price of the asset wherever one finds the forward price of the stock in the Black-Scholes equation. For example, for a European call on a dividend paying stock, we use

$$d_1{*} = \frac{\ln[(Se^{r(T-t)} - FVD) / K] + .5s^2(T-t)}{s\sqrt{T-t}}$$

where FVD is the future value of the dividend on the stock. As a result, we can write the price of a European call on a dividend paying stock as:

$$c = e^{-r(T-t)}\{[e^{r(T-t)}S_t - FVD]N(d_1*) - KN(d_2*)\}$$

We can use similar reasoning to price options on other assets. For example, consider an option on a futures or forward contract. The forward price of the futures (or forward) price is just the futures (forward) price itself. Therefore, if the futures price equals $F_{t,T}$ then the price of a call option on this future is:

$$c = e^{-r(T-t)}[F_{t,T} N(d_1') - KN(d_2')]$$

where

$$d_1' = \frac{\ln(F_{t,T} / K) + .5s^2(T-t)}{s\sqrt{T-t}}$$

This formula is called the "Black Model."

Finally, consider the value of an option on a foreign currency. If the foreign interest rate is r_F and the spot price of the currency is S_t then

$$c = e^{-r(T-t)}[e^{(r-r_F)(T-t)} S_t N(d_1'') - KN(d_2'')]$$

where

$$d_1'' = \frac{\ln(S_t / K) + (r - r_F + .5s^2)(T - t)}{s\sqrt{T - t}}$$

DYNAMIC HEDGING: DELTA, GAMMA, AND VEGA

The Black-Scholes model also describes how the price of the option changes when certain underlying variables change. For example:

$$\frac{\partial c}{\partial S_t} = N(d_1) \equiv \Delta$$

The interpretation for this "Delta" is exactly the same as the interpretaion of the Δ in the binomial model: It gives the number of shares of stock to hold with a short call position in order to form a riskless position. To see why, note that if one holds Δ shares of stock and one short call, the change in the value of the portfolio is

$$\frac{\partial(-c + \Delta S_t)}{\partial S_t} = -N(d_1) + N(d_1) = 0$$

Note that the value of this position doesn't change even if the stock price changes. Thus, it is riskless.

Put differently, if one wants to hedge a single long call, one *sells* Δ shares of the stock.

It is possible to show that if one changes Δ continuously in response to changes in the stock price, one's position will be riskless. This is called "dynamic hedging." It is necessary to change Δ every instant for this strategy to work perfectly.

Of course, in the real world, it is impossible to make such adjustments. Moreover, transactions costs make this strategy prohibitive. Therefore, it is necessary to make adjustments less frequently.

It is desirable to know, however, when it is necessary to make changes most frequently. The "Gamma" of an option provides this information.

Formally,

$$\Gamma = \frac{\partial^2 c}{\partial S_t^2} = \frac{\partial \Delta}{\partial S_t}$$

That is, Γ tells you how rapidly Δ changes when the stock price changes. It quantifies the curvature of the option price function.

This is important information for a trader managing the risk of an option position. Left untended, a position with substantial $\Gamma>0$ may become overhedged (i.e., the trader's actual stock position becomes substantially larger than the optimal Δ) or may become unerhedged (i.e., the traders actual stock position becomes substantially smaller than the optimal Δ) in response to even small changes in the stock price. Put differently, you are likely to need to change a hedge position quickly if there is a lot of Gamma. Gamma neutral hedgers (i.e., hedgers with positions with Γ=0) can sleep better than hedgers with substantial Gamma (either positive or negative).

The other important variable a hedger needs to track is Vega. This gives the sensitivity of the option price to changes in σ. That is,

$$\Psi = \frac{\partial c}{\partial \sigma}$$

Remember that Black-Scholes assumes that volatility is constant over time. In reality, however, volatility can change rapidly. If so, a trader with substantial Vega is subject to large gains or losses. Just as a hedger wants to know his vulnerability to stock price moves, he should also track his risk to volatility changes.

These "Greeks" are important for another reason. Consider a trader who wants to purchase or sell an option that is not traded on an organized exchange, or on the OTC market. (For example, an individual might want to buy a 5 year call on the S&P 500). Even though the trader cannot trade the option, he can replicate the payoffs of the option synthetically by dynamically buying or selling the asset underlying the option.

That is, at every instant of time, the trader holds Δ units of the asset underlying the option, where Δ gives the delta of the option he wants to replicate. If the trader can adjust the position continuously and costlessly, such a strategy will produce cash flows at option expiration that are exactly identical to those of the option.

You may also want to replicate an option that is traded. For example, if you want to buy a put, but you think puts are overpriced in the market (e.g., the implied volatility of traded puts is greater than your forecast of volatility), you may wish to replicate the put through a dynamic trading strategy. Similarly, you can arbitrage the market by buying (selling) underpriced (overpriced) options, and dynamically replicating off-setting positions.

In reality, it is impossible (and very costly!) to adjust the portfolio continuously in this fashion. This is particularly true if the option is near the money, so has a lot of Gamma. Moreover, the dynamic hedger is vulnerable to changes in volatility. The problems faced by "portfolio insurers" (traders who attempted to replicate calls on the stock market through dynamic hedging techniques) on 10/19/87 provides a graphic illustration of the potential problems.

In order to address these problems, the trader can add traded options to his portfolio. The trader's objective is to match the Delta, Gamma, and Vega of his portfolio to the Delta, Gamma, and Vega of the option the trader desires to replicate. This reduces the amount of adjustment needed, and provides some insurance against volatility shocks.

For example, consider replicating an option with $\Delta=\Delta^*$ and $\Gamma=\Gamma^*$. You can use the underlying stock and an option with $\Delta=\Delta^{**}$ and $\Gamma=\Gamma^{**}$ to construct your replicating portfolio. To get a delta and gamma match, you have two equations and two unknowns. The unknowns are N_S, the number of shares of stock to trade, and N_o, the number of options to buy or sell as part of your hedge portfolio. Formally, solve:

$$\Delta^* = N_S + N_o \Delta^{**}$$

and

$$\Gamma^* = N_o \Gamma^{**}$$

You could also construct a delta, gamma, and vega matched position. To do so require two options and the stock. You have to solve three equations in three unknowns here.

Note that even a gamma matched or gamma and vega matched position must be adjusted over time. Recently, researchers have developed strategies that allow you to create “fire-and-forget” hedges. That is, using these techniques, you can replicate an option by constructing a portfolio at the initiation of the trade, and never adjusting the portfolio until the expiration date of the option you want to replicate. As you might guess, this requires you to trade in a large number of options.

FORMULAE FOR GAMMA, VEGA, AND THETA

(The P in the formulae is the mathematical constant 3.14159)

The Gamma of a Euro call or put equals:

$$\Gamma = \frac{e^{-.5d_1^2}}{Ss\sqrt{2p(T-t)}}$$

The Vega of a Euro call or put equals:

$$\Lambda = S\sqrt{T-t}\,\frac{e^{-.5d_1^2}}{\sqrt{2p}}$$

The Theta of a Euro call equals:

$$\Theta = -\frac{Se^{-.5d_1^2}\,s}{2\sqrt{2p(T-t)}} - rKe^{-r(T-t)}N(d_2)$$

The Theta of a Euro put equals:

$$\Theta = -\frac{Se^{-.5d_1^2}\,s}{2\sqrt{2p(T-t)}} + rKe^{-r(T-t)}N(-d_2)$$

Note that Theta is unambiguously negative for a Euro call, but may be positive for a Euro put.

For a call, Gamma, Delta, and Theta are related by this equation:

$$rc = \Theta + rS\Delta + .5\, s^{2} S^{2} \Gamma$$

A similar expression holds for a put.

FORMULAE FOR GAMMA, VEGA, AND THETA
On FORWARDS
(The P in the formulae is the mathematical constant 3.14159)

The Delta of a Euro futures call equals:

$$\Delta = e^{-r(T-t)} N(d_1^{'})$$

The Delta of a Euro futures put equals:

$$\Delta = -e^{-r(T-t)} N(-d_1^{'})$$

Gamma of a Euro futures call or put equals:

$$\Gamma = \frac{e^{-r(T-t)} e^{-.5d_1^2}}{Fs\sqrt{2p(T-t)}}$$

The Vega of a Euro call or put equals:

$$\Lambda = F\sqrt{T-t}\,\frac{e^{-r(T-t)} e^{-.5d_1^2}}{\sqrt{2p}}$$

The Theta of a Euro futures call equals:

$$\Theta = e^{-r(T-t)}[-\frac{Fe^{-.5d_1^2}s}{2\sqrt{2p(T-t)}} - rFN(-d_1') - rKN(d_2')]$$

The Theta of a Euro futures put equals:

$$\Theta = e^{-r(T-t)}[-\frac{Fe^{-.5d_1^2}s}{2\sqrt{2p(T-t)}} - rFN(-d_1') + rKN(-d_2')]$$

Note that Theta is unambiguously negative for a Euro call, but may be positive for a Euro put.

For a call, Gamma, Delta, and Theta are related by this equation:

$$rc = \Theta + .5s^2 F^2 \Gamma$$

A similar expression holds for a put.

Using Black-Scholes

- Most of the inputs to the Black-Scholes model are observable. These include: the strike price, the underlying price, the interest rate, and time to expiration. One key input is **not** observable—the volatility.

- Good option values depend on good volatility values.

- Where do volatility values come from? There are two sources: historical data and implied values.

- Historical volatilities are calculated using historical data on returns. For instance, to calculate the volatility for MSFT stock, one would plug historical data on MSFT returns into a spreadsheet and calculate the standard deviation (adjusted for the data frequency).

- If the B-S model were strictly correct, this approach would be the right one. Moreover, if the B-S model were strictly correct—that is, if volatility was truly constant—you would want to get as much historical data as is available to get the most precise estimates of volatility possible.

- An alternative approach is to use "implied volatility." In this approach, you find the value of σ that sets the value of the option given by the B-S formula equal to the value of the option observed in the marketplace.

- Implied volatility can be estimated with extreme accuracy using the "Solver" function of Excel. There are very good approximations that you can use to determine the implied volatility for an at-the-money option.

- Obviously, if one uses the implied volatility, one is assuming that (a) the model is correct, and (b) the option is correctly priced. Thus, implied volatility cannot be used to determine whether a particular option is mis-priced. So what is it good for?

- If the B-S model is correct, all options should have the same implied vol. Therefore, comparing implied volatilities across options allows one to determine whether options on the same underlying are mis-priced relative to one another. Buy the option with low volatility, sell the option with high volatility.

- Also, you can use the implied volatility to get more accurate measures of the "Greeks."

- In practice, most options are quoted using an implied volatility (that is then plugged into the B-S formula). Therefore, most options traders think in vol terms rather than in price terms, and option trading is essentially volatility trading.

How Well Does the Black-Scholes Model Work?

- The Black-Scholes model is the most influential theory ever introduced in finance, and perhaps in all of the social sciences. Despite this success, it has its weaknesses. Practitioners have learned how to patch the holes in Black-Scholes.

- We know that the B-S model is not strictly correct. First, contrary to the model's predictions, there are systematic differences in implied volatility by strike price and time to expiration.

- Most important, deep in-the-money and deep out-of-the-money options have consistently higher volatilities than at-the-money options. This is referred to as the "volatility smile." For equity index options, this is more of a "smirk" because options with low strike prices have far higher implied vols than ATM or high strike options.

- This implies that σ cannot be constant.

- This is confirmed by empirical evidence. This evidence shows that volatility varies systematically over time. Moreover, volatility "clumps"—big price moves tend to follow big price moves, and small price moves tend to follow small price moves.

Dealing With the Smile

- There are two basic approaches to the smile.
- The first approach posits that volatility is a function of the underlying price and time. That is, $s\,(S,t)$. Using very advanced techniques, one can find such a function that fits a variety of options prices. (Many practitioners use a crude approach that gives a very unreliable and unstable estimate of this function).
- The second approach is to post that volatility is random and to write down a specific stochastic process for volatility.
- The second approach is probably more realistic, but raises many difficulties. Most important, volatility risk cannot be hedged using the underlying. Consequently, any pricing formula must include a volatility risk premium.
- Some practitioners and academics assume that this risk premium is zero. This results in convenient pricing formulae, but is inconsistent with a great deal of evidence. Most notably, options hedged against moves in the underlying price earn a positive risk premium—this wouldn't happen if vol risk premia equal zero.
- Thus, the smile is a knotty issue that most practitioners address in an *ad hoc* manner.

How Exotic

- So far we have considered "vanilla" options—basic puts and calls.
- Other kinds of options—"exotics"—are traded in the OTC market.
- Although the variety of exotics is limited only by the imagination of traders, some exotics are more common than others.
- Sometimes exotics are bundled implicitly in securities.

Digital or “Bet” Options

- A digital option pays a fixed amount of money if a certain event occurs.

- For instance, a digital call on Microsoft struck at $75 that pays $10, pays $10 if the price of MSFT at expiration exceeds $75, regardless of whether the price at expiration is $75.01 or $175. Similarly, a digital put struck at $75 that pays $10, pays $10 if the price of the underlying at expiration is $74.99 or $0.

- The value of a digital is easy to determine. A digital call value is $e^{-r(T-t)}PN(d_2)$ where P is the payoff and d_2 is the same as in the B-S formula. Similarly, a digital put value is $e^{-r(T-t)}PN(-d_2)$.

- Digitals are traded in the OTC market, but perhaps the most (in)famous example of digital options was embedded in bonds bought by Orange County CA in the early 1990s. To raise yields the Orange Cty treasurer bought bonds that had embedded short digital options on interest rates.

- A “one touch” option is an American digital. It is called a one touch because it is optimal to exercise as soon as the underlying price hits (“touches”) the strike price (do you know why?)

Knock-Options

- Knock options come in several varieties.

- Knock-in options. These are options with an underlying option that comes into existence only if some condition is met. There are up-and-in and down-and-in varieties. For example, an up-and-in call with a strike price of $75 and a "knock barrier" of $100 has a payoff at expiration if-and-only-if the underlying price hit or exceeded $100 some time prior to expiration.

- Knock-out options. These are options that go out of existence if some condition is met. For example, a down-and-out call struck at $75 with a knock barrier of $50 expires worthless if the underlying price hits or is less than $50 at any time during the option's life. Thus, even if the stock price is $100 at expiration, the option is worthless if the stock price hit $50 prior to expiration.

- Knock-options are "path dependent" because their payoff depends not only on the value of the underlying at expiration, but the path that the underlying price follows prior to expiration.

- Manipulation can be an issue with these "barrier options."

Asian Options

- An Asian option has a payoff that depends on the average price of the underlying over some time period.

- For instance, an Asian call on crude oil with a monthly averaging period has a payoff that depends on the average price of crude oil over a one-month-long period.

- There are average price options where the strike price is set prior to expiration, and the payoff is based on the difference between an average price and the pre-agreed strike price.

- There are also "average strike" options where the strike price is the average price on the underlying during some averaging period, and the payoff is based on the difference between the value of the underlying at expiration and this strike price.

- There are no closed form solutions for Asian options with arithmetic averaging. Various numerical techniques can be used to price them.

Some Other Exotics

- Compound options. These are options on an option, such as a put on a call, or a call on a call. There are formulae for valuing such options given Black-Scholes assumptions.

- Exchange options. These give you the right, but not the obligation to exchange one asset for another. There are formulae for valuing such options given Black-Scholes assumptions.

- "COD" (cash-on-delivery) options. Here you only pay the premium if they wind up in the money. These options can have a negative payoff.

- Quantos. Cross-currency options are the most common example. An example is a contract on the Nikkei Index with the payoff converted to USD. This conversion can occur at either an exchange rate set when the option is written or the exchange rate prevailing at expiration. The payoff to the option is $X\max[S\text{-}K,0]$ where X is the exchange rate, S is the value of the Nikkei, and K is the strike price (in JY). Here there are two sources of risk—the Nikkei index (in yen), and the JY-USD exchange rate.

- Compos. These are options with payoffs $\max[XS\text{-}K,0]$ where the strike price is in the domestic currency.

- Lookbacks. Lookbacks are path dependent options because the payoff depends on the maximum (or minimum) price reached by the underlying during the option's life. For instance, a lookback call with a strike price K has a payoff $\max[\max(S)-K,0]$ where $\max(S)$ is the maximum priced achieved by the underlying during the life of the option. Obviously this lookback is more valuable than a vanilla call.

Issues With Exotics

- Exotics frequently present serious hedging difficulties, especially when they are near the money and close to expiration.

- For example, a digital option's gamma is positive when it is out of the money and negative when it is the money. Right before expiration, the option's gamma is positive infinity at a price immediately below the strike price and negative infinity at a price immediately above the strike price. This behavior makes it hard to hedge.

- As another example, a knock-out option has an infinite gamma when the underlying price is at the knock-out boundary.

- Compound options have high gammas when they are at the money because they are options on options—the convexity in the underlying option compounds the convexity in the "top" option.

- Recall that convexity also has ramifications for the sensitivity option value to volatility. Hence, some exotics may have substantial vol (vega) risk.

Get Real

- Heretofore we have discussed options that are financial claims.
- The world is full of non-financial—“real”—options.
- In particular, virtually all business investment and operational decisions involve choice—i.e., optionality.
- Option to defer investment.
- “Time-to-build” option—build a project in stages with the choice to abandon after each stage.
- Option to increase or reduce investment scale.
- Option to abandon.
- Option to switch inputs or outputs (operational flexibility).
- Growth options (R&D, leases on property or resources).
- Combinations of the above options.

Valuing Real Options

- Traditional capital budgeting approaches ignore real options. This is unfortunate as real options may have a large affect on the profitability of investment.
- Firms may underinvest, overinvest, or invest at the wrong time if they ignore real options.
- Increasingly firms are using option pricing techniques to evaluate investment projects.
- Given the intense informational demands, it is often not practical to use these techniques to get precise estimates of real option value.
- Nonetheless, the use of options valuation techniques forces firms to analyze the characteristics of their investments more rigorously.

An Example

- Consider a firm that has the option to defer investment in drilling a gas well. It can invest today, spend $104, and get a well worth $100 (i.e., which generates $100 in net present value). From an NPV perspective, this project is a turkey.

- However, the investment is risky. In one year, if good news about the value of gas arrives, the well is worth $180. If bad news arrives, it is worth $60. The interest rate is 8 percent per year.

- In terms of our binomial model, u=1.8 and d=.6. The "pseudo-probability" is (1.08-.6)/(1.8-.6)=.4.

- The option to delay the project one year is therefore [.4(180-104)+.6(0)]/1.08=28.15

- Exercise: Show that the value to delay the project for up to two years is worth $31.82, and the value to delay the project for up to three years is worth $45.01.

Intuition

- The intuition here is pretty straightforward—the option to delay allows you to wait for the arrival of information, namely whether gas prices are going to rise or fall. Since better information means better investment decisions, you the option to delay is valuable.

- The simple example here implies that you will never invest if you always have the option to delay. This is an American call option. Under the assumptions of the analysis (with no intermediate cash flows) standard analysis implies that deferral is always the best choice.

- In the real world, investments throw off benefits only if the firm invests in the asset. These benefits are like dividends and can induce the firm to exercise the call option early.

- In commodity markets, for instance, prices tend to be "mean reverting"—that is, if the price is higher (lower) than average today, it is expected to fall (rise). This mean reversion can make it optimal to invest now rather than waiting.

Implications

- Real options have important implications.

- Under some circumstances, the value of investment options is increasing in uncertainty (i.e., the volatility of the value of the asset).

- An increase in uncertainty induces firms to defer investment.

- Policy uncertainty (e.g., uncertainty in tax, regulatory or monetary policy) can affect firm investment strategies. One would expect to see greater levels of investment in “stable” jurisdictions (countries, states) than “unstable” ones